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CERTAIN FRACTIONAL CALCULUS OPERATORS ASSOCIATED WITH FOX-WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT

In this paper Riemann-Lioville fractional integral and derivative formulas for Fox-Wright ${}_p\varphi_q(z)$ generalized hypergeometric functions are obtained. Some special cases of the established formulas are also discussed.

2000 Mathematics Subject Classification : 26A33, 45D05.

Keywords : Fox-Wright generalized hypergeometric Function, Riemann-Liuville fractional Integral and differential.

1. Introduction. The Fox-Wright generalized hypergeometric ${}_p\varphi_q(z)$ for $z \in C$ is defined in series form as [4].

$$ {}_p\varphi_q(z) = {}_p\varphi_q\left[\left.\begin{matrix}(a_i, A_i)_{1,p}\\(b_i, B_j)_{1,q}\end{matrix}\right| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_j k) z^k}{\prod_{i=1}^{q}\Gamma(b_i + B_j k) k!} \qquad ...(1.1)$$

where $a_j, b_j \in C, A_i > 0, B_j > 0$

$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_i \geq 0; A_i, B_j \in R(A_i, B_j \neq 0)(i = 1,...,p, j = 1,...,q)$ for suitably bounded value of $|z|$.

As special case, for $A_1 = ... = A_p = 1$, $B_1 = ... = B_q = 1$, (1.1) reduces to generalized hypergeometric function [3]

$$ {}_p\phi_q\left[\left.\begin{matrix}(a_1,1),...,(a_p,1)\\(b_1,1),...,(b_q,1)\end{matrix}\right| z\right] = \frac{\Gamma(a_1),...,\Gamma(a_p)}{\Gamma(b_1),...,\Gamma(b_q)}\, {}_pF_q\left[\begin{matrix}a_1,...,a_p;\\b_1,...,b_q;\end{matrix}\, z\right] \qquad ...(1.2)$$

For $p=1, q=1, a_1=\delta, A_1=1, b_1=\beta$ and $B_1=\alpha$, (1.1) reduces to

$$ {}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\beta,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(\delta+k)}{\Gamma(\beta+\alpha k)}\frac{z^k}{k!}=\Gamma(\delta)E_{\alpha,\beta}^{\delta}(z) \qquad ...(1.3)$$

called generalized Mittag-Leffler function defined by Prabhakar [2].

If $\delta=1$ then (1.3) reduces to generalized Mittag-Leffler function

$$ {}_1\varphi_1\left[\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(1+k)}{\Gamma(\beta+\alpha k)}\frac{z^k}{k!}=E_{\alpha,\beta}(z).$$

If $\delta=1$ and $\beta=1$ then (1.3) reduces to

$$ {}_1\varphi_1\left[\begin{matrix}(1,1)\\(1,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(1+k)}{\Gamma(1+\alpha k)}\frac{z^k}{k!}=E_{\alpha}(z) \qquad ...(1.3)$$

called Mittag-Leffler function [1].

The object of this paper is to derive the relations which exist between the Fox-Wright generalized hypergeometric function ${}_p\varphi_q(z)$ and left and right side Riemann-Liouville fractional integral and differential operators.

The fractional integral and differential operators defined by Samko, Kilbas and Marichev [7] for $\alpha>0$ are given by

$$\left(I_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, \qquad ...(1.4)$$

$$\left(I_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)dt, \qquad ...(1.5)$$

$$\left(D_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x(x-t)^{-\{\alpha\}}f(t)dt, \qquad ...(1.6)$$

$$\left(D_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty}(t-x)^{-\{\alpha\}}f(t)dt, \qquad ...(1.7)$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

2.Properties of generalized Fox-Wright hypergeometric function. In this section we derive several interesting properties of the generalized Fox-

Wright hypergeometric function ${}_p\varphi_q(z)$ defined by (1.1) with the help of Riemann -Liouville fractional integral and derivative formula [4].

Theorem 1. Let $\alpha>0,\beta>0,\gamma>0$ and $a\in R$, let I_{0+}^{α} be left sided operator of Riemann-Liouville fractional integral (1.4). Then there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_p\varphi_q\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{\beta}\right]\right\}(x)=x^{\gamma+\alpha-1}{}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_1)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\right|ax^{\beta}\right]\quad\text{...(2.1)}$$

Proof. By virtue of (1.1) and (1.4) we have

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_p\varphi_q\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{\beta}\right]\right\}(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{(x-t)^{\alpha-1}a^kt^{\beta k+\gamma-1}}{k!}dt\,.$$

Interchanging the order of integration and summation; and evaluating the inner integral with the help of beta functions, by setting $t=xy$, we get

$$L.H.S.=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_i+B_jk)}\frac{a^kx^{\beta k+\gamma+\alpha-1}}{\Gamma(\alpha)k!}\int_0^1(1-y)^{\alpha-1}y^{\beta k+\gamma-1}dy$$

$$=x^{\gamma+\alpha-1}{}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_i,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\right|ax^{\beta}\right].$$

Interchanging the order of integration and summation is permissible under the conditions stated with the theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 1.

Corollary 1.1 .For $\alpha>0,\beta>0,\gamma,\lambda>0$, there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_1\varphi_1\left[\left.\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\right|at^{\beta}\right]\right\}(x)=\Gamma(\delta)x^{\gamma+\alpha-1}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{\beta}\right)\quad\text{...(2.2)}$$

Corollary 1.2 . By setting $\delta=1$ in (2.2) there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}E_{\beta,\alpha}\left(at^{\beta}\right)\right\}(x)=x^{\gamma+\alpha-1}E_{\beta,\alpha+\gamma}\left(ax^{\beta}\right)\quad\text{...(2.3)}$$

establised by Saxena and Saigo [6].

Theorem 2. Let $\alpha>0,\beta>0,\gamma>0$ and $a\in R$, let I_{-}^{α} be right operator of Riemann Liouville fractional integral (1.5). Then there holds the formula

$$I_-^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| at^{-\beta}\right]\right\}(x)=x^{-\gamma}\ {}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle| ax^{-\beta}\right] \quad ...(2.4)$$

Proof. By virtue of (1.1) and (1.5), we have

$$I_-^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| at^{-\beta}\right]\right\}(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{(t-x)^{\alpha-1}\,a^kt^{-\beta k-\alpha-\gamma}}{k!}dt.$$

Interchanging the order of integration and summation and then evaluating the inner integral by beta function formula, we get

$$L.H.S.=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_i+B_ik)}\frac{a^k}{\Gamma(\alpha)k!}\int_x^{1}(t-x)^{\alpha-1}t^{-\beta k-\alpha-\gamma}dt.$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{\Gamma(\alpha)k!}\int_0^{\infty}y^{\alpha-1}(x+y)^{-\beta k-\alpha-\gamma}dt \quad \text{(by putting } t-x=y\text{)}$$

$$=x^{-\gamma}\ {}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle| ax^{-\beta}\right] \quad \text{(putting } y=wx\text{)}.$$

Interchanging the order of integration and summation is permissible under the conditions stated with the theorem, due to convergence of the integrals involved in the process. This completes the proof of Theorem 2.

Corollary 2.1. For $\alpha>0,\beta>0,\gamma>0,a\in R$, there holds the formula

$$I_-^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle| at^{-\beta}\right]\right\}(x)=\Gamma(\delta)x^{-\gamma}E^{\delta}_{\beta,\alpha+\gamma}(ax^{-\beta}). \quad ...(2.5)$$

Corollary 2.2. By setting $\delta=1$ in (2.5), there holds the formula

$$I_-^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_1\varphi_1\left[\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle| at^{-\beta}\right]\right\}(x)=x^{-\gamma}E_{\beta,\alpha+\gamma}(ax^{-\beta}) \quad ...(2.6)$$

estabilished by Saxena and Saigo [6].

Theorem 3. Let $\alpha>0,\beta>0,\gamma,\lambda>0$, $a\in R$ and D_{0+}^{α} be left sided operator of Riemann Liouville fractional integral (1.6). Then there holds the formula

$$D_{0+}^{\alpha}t^{\gamma-1}\,{}_1\varphi_1\left[t^{\gamma-1}\,{}_p\varphi_q\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{\beta}\right]\right](x)=x^{\gamma-\alpha-1}\,{}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha-\gamma,\beta)\end{matrix}\right|ax^{\beta}\right]\quad\text{...(2.7)}$$

Proof. By virtue of (1.1) and (1.6), we have

$$D_{0+}^{\alpha}t^{\gamma-1}\,{}_p\varphi_q\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{\beta}\right](x)=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}t^{\gamma-1}\,{}_p\varphi_q\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{\beta}\right](x)\right)$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{\Gamma(1-\{\alpha\})k!}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x(x-t)^{1-\{\alpha\}-1}t^{\beta k+\gamma-1}dt.$$

Interchanging the order of integration and summation and evaluating the inner integral by beta function formula (by setting $t=xy$), we obtain

$$L.H.S.=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{k!}\frac{\Gamma(\gamma+\beta k)}{\Gamma(\gamma+\beta k+1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}x^{\beta k+\gamma-\{\alpha\}}$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{\Gamma(\gamma+\beta k)}{\Gamma(\gamma+\beta k-\alpha)}\frac{x^{\beta k+\gamma-\alpha-1}a^k}{k!}$$

$$=x^{\gamma-\alpha-1}\,{}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha-\gamma,\beta)\end{matrix}\right|ax^{\beta}\right].$$

Interchaning the order of integration and summation is permissible under the condition stated with the theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 3.

Corollary 3.1. For $\alpha > 0, \beta > 0, \gamma, \lambda > 0$, there holds the formula

$$D_{0+}^{\alpha}\left\{t^{\gamma-1}\,{}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right]\right\}(x) = \Gamma(\delta)x^{\gamma-\alpha-1}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{\beta}\right) \quad \text{...(2.8)}$$

Corollary 3.2. By setting $\delta = 1$ in (2.8) then there holds the formula

$$D_{0+}^{\alpha}\left\{t^{\gamma-1}\,{}_1\varphi_1\left[\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right]\right\}(x) = x^{\gamma-\alpha-1}E_{\beta,\alpha+\gamma}\left(ax^{\beta}\right) \quad \text{...(2.9)}$$

established by Saxena and Saigo [5].

Theorem 4. Let $\alpha > 0, \beta > 0, \gamma, \lambda > 0$, $a \in R$ and D_{-}^{α} be right sided operator of Riemann Liouville fractional integral (1.7). Then there holds the formula

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| at^{-\beta}\right]\right\}(x) = x^{-\gamma}\,{}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle| ax^{-\beta}\right] \quad \text{...(2.10)}$$

Proof. By virtue of (1.1) and (1.7), we have

$$D_{-}^{\alpha}t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| at^{-\beta}\right](x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle| at^{\beta}\right](x)\right)$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{\Gamma(1-\{\alpha\})k!}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty}(t-x)^{-\{\alpha\}}t^{-\beta k+\alpha-\gamma}dt$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{k!\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^{\infty}(x+y)^{-\beta k+\alpha-\gamma}y^{1-\{\alpha\}-1}dy$$

(putting $t-x=y$)

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k\Gamma(\beta k+\gamma-[\alpha]-1)}{k!\Gamma(\beta k+\gamma-\alpha)}\left(\frac{d}{dx}\right)^{[\alpha]+1}x^{-\beta k-\gamma+[\alpha]+1} \quad \text{(putting } y=wx\text{)}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_j + B_j k)} \frac{a^k \Gamma(\beta k + \gamma)}{k!\, \Gamma(\beta k + \gamma - \alpha)} x^{-\beta k - \gamma}$$

$$= x^{-\gamma}\ {}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i, A_i)_{1,p}, (\gamma, \beta) \\ (b_j, B_j)_{1,q}, (\alpha - \gamma, \beta)\end{matrix}\right| a x^{-\beta}\right].$$

Interchanging the order of integration and summation is permissible under the condition stated with the Theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 4.

Corollary 4.1. For $\alpha > 0, \beta > 0, \gamma, \lambda > 0$, there holds the formula

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\ {}_1\varphi_1\left[\left.\begin{matrix}(\delta, 1) \\ (\gamma, \beta)\end{matrix}\right| a t^{-\beta}\right]\right\}(x) = \Gamma(\delta) x^{-\gamma} E_{\beta, \alpha-\gamma}^{\delta}\left(a x^{-\beta}\right) \qquad \text{...(2.11)}$$

Corollary 4.2 . For $\delta = 1$, (2.11) takes the form

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\ {}_1\varphi_1\left[\left.\begin{matrix}(1, 1) \\ (\gamma, \beta)\end{matrix}\right| a t^{-\beta}\right]\right\}(x) = x^{-\gamma} E_{\beta, \alpha-\gamma}\left(a x^{-\beta}\right) \qquad \text{...(2.12)}$$

established by Saxena and Saigo [6].

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LIE THEORY AND BASIC GAUSS POLYNOMIALS

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ABSTRACT

In the Present paper, an attempt has been made to bring basic hypergeometric functions within the purview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function ${}_2\phi_1$. Multiplier representation theory is then used to obtain generating function for basic analogue of Gauss polynomial. The results obtained in this paper are extensions of the results derived earlier by Miller [3] and Sarkar-Chatterjea [4].

2000 Mathematics Subject Classification : Primary 60E07, 20G05; Secondary 33C55, 14A17.

Keywords : Lie algebra, Generating function, Basic Gauss polynomials.

1. Introduction. The q-analogue of the Gauss functions or Heine's series [1] may be written as

$$ {}_2\phi_1(a,b;c;q;x)=\sum_{n=0}^{\infty}[a;q,n][b;q,n]/[c;q,n][n;q]! \quad (c\neq 0,-1,-2,\ldots)$$

where $|q|<1$ and $|x|<1$.

Here $[a;q,n]$ and $[n;q]!$ are respectively the basic Pochammer's symbol and basic factorial function defined as $[a;q,n]=[a;q][a+1;q]\ldots[a+n-1;q]$ and $[n;q]!=[1;q][2;q]\ldots[n;q]$.

The basic differential operator $B^{\wedge}_{q,x}$ is defined by [1] through the relation

$$B^{\wedge}_{q,x}\phi(x)=\{\phi(qx)-\phi(x)\}/x(q-1). \quad \ldots(1.1)$$

2. The Dynamical Symmetry Algebra of ϕ. The dynamical symmetry

algebra of the hypergeometric function has been defined by Miller [2]. We use the same technique to define the dynamical symmetry algebra of ${}_2\phi_1$. Let

$$\phi_{\alpha,\beta,\gamma,q} = \Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma).{}_2\phi_1[\alpha,\beta;\gamma;q;x]s^{\alpha}u^{\beta}t^{\gamma} \quad ...(2.1)$$

be the basis elements of a subspace of analytical functions of four variables x,s,u and t, associated with Heine's basic hypergeometric function of Heine's series ${}_2\phi_1$. Introduction of variables s,u and t renders differential operators independent of parameters α,β and γ and thus facilitates their repeated operation.

The dynamical symmetry algebra of ${}_2\phi_1$ is a 15-dimensional complex Lie algebra isomorphic to sl (4), generated by twelve $E^{\wedge}$-operators termed as raising or lowering operators in view of their effect of raising or lowering the corresponding suffix in $\phi_{\alpha\beta\gamma,q}$. The $E^{\wedge}$-operators are

(i) $E^{\wedge}_{-\alpha,q} = s^{-1}\left(x(1-x)B^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - sB^{\wedge}_{q,s} - xuB^{\wedge}_{q,u}\right),$

(ii) $E^{\wedge}_{-\beta,-\gamma,q} = u^{-1}t^{-1}\left(x(1-x)B^{\wedge}_{q,x} - xsB^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - 1\right),$ ...(2.2)

The action of these operators on $\phi_{\alpha\beta\gamma,q}$ is given by

$$E^{\wedge}_{-\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha-1;q]\phi^{-1}_{\alpha,q,\beta,\gamma,q}$$

$$E^{\wedge}_{-\beta,-\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma-\alpha-1;q]\phi^{-1\ \ -1}_{\alpha,\beta,\gamma,\gamma,q} \quad ...(2.3)$$

The upper factor in each bracket is to be associated with plus sign and lower with minus sign. Twelve E-operators together with three maintenance operators $J_{\alpha}, J_{\beta}, J_{\gamma}$ and Identity operator I form a basic for $gl(4) \cong sl(4)(I)$, where (I) is the 1-dimensional Lie algebra generated by 1.

Here

$$J^{\wedge}_{\alpha,q} = sB^{\wedge}_{q,s} J^{\wedge}_{\beta,q} = uB^{\wedge}_{q,u}, J^{\wedge}_{\gamma,q} = tB^{\wedge}_{q,t} \text{ and } I^{\wedge} = 1 \quad ...(2.4)$$

with the results

$$J^{\wedge}_{\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\beta,q}\phi_{\alpha,\beta,\gamma,q} = [\beta;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma;q]\phi_{\alpha,\beta,\gamma,q},$$

and

$$I^{\wedge}\phi_{\alpha,\beta,\gamma,q} = \phi_{\alpha,\beta,\gamma,q}. \quad ...(2.5)$$

3. The Generating Functions for Basic Analogues of Gauss Polynomials. On comparing the results obtained by the action one parameter subgroup $\left(exp_q\, aE^{\wedge}_{-\alpha,-\gamma,q}\right)$ generated by the operator $E^{\wedge}_{-\alpha,q}$ defined in (2.2) on $\phi_{\alpha,\beta,\gamma,q}$ defined in (2.1) and direct expansion, we get the identity

$$\left[st/(ax+st)\right]^{\beta}\left[a/(st+1)\right]^{\gamma-1}\ {}_2\Phi_1\left[\alpha,\beta;\gamma;q;x(a+st)/(ax+st)\right]$$

$$=\sum_{m=0}^{\infty} a^{m}\left[\gamma-m;q\right]_m/[m;q]!\ {}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q^{-m};q;x\right](st)^{-m} \quad ...(3.1)$$

Taking, $\alpha\to 0,\beta\to\lambda+\mu+m-1,\gamma\to q^{\lambda+m},st\to 1,a\to 1,$ we get

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}[\gamma;q]_m/[m;q]!.\,{}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right] \quad ...(3.2)$$

By definition of basic Gauss polynomial [1]

$G_m^{\lambda,\mu}(q;x)={}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right],$

where $\lambda\neq 0,-1,-2,3,...$ . ...(3.3)

Using (3.3) in (3.2), we get the generating function

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}[\gamma;q]_m/[m;q]!.G_m^{\lambda,\mu}(q;x)$$

for basic Gauss polynomials.

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RECURRENCE RELATIONS FOR THE $\bar{H}$-FUNCTION

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ABSTRACT

The aim of the present paper is to establish two new recurrence relations for the $\bar{H}$-function. Some results for the generalized Wright hypergeometric function and generalized Riemann Zeta function are special cases of our main findings.

2000 Mathematics Subject Classification : 33C60

Keywords and Phrases : Generalized Wright Hypergeometric Function, Generalized Riemann Zeta Function and $\bar{H}$-function.

1. Introduction. The $\bar{H}$-function was introduced by Inayat Hussain [4] and studied by Bushman and Srivastava [1].

The $\bar{H}$-function is defined and represented in the following manner:

$$\bar{H}_{P,Q}^{M,N}[z]=\bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_L\bar{\phi}(\xi)z^{\xi}d\xi,(z\neq 0) \qquad ...(1.1)$$

where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad ...(1.2)$$

The behaviour of $\bar{H}$ -function for small valkues of $|z|$ follows easily from a result recently given by Rathie {[6],p.306, eq.(6.9)}.

2. Main Results. In this section, we establish two recurrence relations.

First Recurrence Relation

$$\bar{H}^{M,N}_{P,Q}[z] = \bar{H}^{M,N}_{P,Q}\left[z \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right]$$

$$= \frac{1}{2\pi\omega}\left\{ e^{\omega\pi b_{M+1}}\bar{H}^{M,N}_{P,Q}\left[ze^{-\omega\pi\beta_{M+1}} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right] \right.$$

$$\left. -e^{\omega\pi b_{M+1}}\bar{H}^{M,N}_{P,Q}\left[ze^{-\omega\pi\beta_{M+1}} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M+1},(b_j,\beta_j,B_j)_{M+2,Q} \end{matrix} \right] \right\} \qquad \ldots(2.1)$$

Second Recurrence Relation

$$\bar{H}^{M,N}_{P,Q}[z] = \bar{H}^{M,N}_{P,Q}\left[z \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right]$$

$$= \frac{1}{2\pi\omega}\left\{ e^{\omega\pi a_{N+1}}\bar{H}^{M,N}_{P,Q}\left[ze^{-\omega\pi a_{N+1}} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+2,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right] \right.$$

$$\left. -e^{\omega\pi a_{N+1}}\bar{H}^{M,N}_{P,Q}\left[ze^{-\omega\pi a_{N+1}} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N+1},(a_j,\alpha_j)_{N+2,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right] \right\} \qquad \ldots(2.2)$$

Recurrence relations (2.1) and (2.2) are valid under the conditions of (1.1) and (1.2).

Proof. The recurrence relations (2.1) and (2.2) can be easily established by appealing to the definition [4], [1] and use the well known relation (Rainville [5], p.21, Th.8)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} = \frac{2\pi\omega}{e^{\omega\pi z} - e^{-\omega\pi z}},$$

involving Γ-function and reinterpret the result in terms of the $\bar{H}$ -function, we arrive at the right hand side of the desire results (2.1) and (2.2).

3. Particulars Cases.

(i) If in the recurrence relation (2.1) we reduce $\bar{H}_{P,Q}^{M,N}$ to generalized Wright hypergeometric function [3, p.271, eq. (7)]

$$
{}_P\Psi_Q\left(\begin{matrix}(a_j;\alpha_j;A_j)_{1,P};\\ (b_j;\beta_j;B_j)_{1,Q};\end{matrix}\; z\right)=\frac{1}{2\pi\omega}\left\{e^{\omega\pi b_{M+1}}\sum_{t=0}^{\infty}\frac{\prod_{j=1}^{P}\{\Gamma(a_j+\alpha_j t)\}^{A_j}}{\prod_{j=M+1}^{P}\{\Gamma(b_j+\beta_j t)\}^{B_j}}\frac{\left(ze^{-\omega\pi\beta_{M+1}}\right)^t}{t!}\right.
$$

$$
\left.-e^{-\omega\pi b_{M+1}}\sum_{t=0}^{\infty}\frac{\prod_{j=1}^{P}\{\Gamma(a_j+\alpha_j t)\}^{A_j}}{\prod_{j=M+1}^{P}\{\Gamma(b_j+\beta_j \xi)\}^{B_j}}\frac{\left(ze^{-\omega\pi\beta_{M+1}}\right)^t}{t!}\right\}.
$$

(ii) If in the recurrence relation (2.2), we reduce $\bar{H}_{P,Q}^{M,N}$ to generalized Riemann Zeta function [3,p.27, eq.(1); 7, p.314-315, eq. (1.6) and (1.7)]

$$
\phi(z,p,\eta)=\frac{1}{2\pi\omega}\left\{e^{\omega\pi a_{N+1}}\bar{H}_{2,2}^{1,2}\left[-ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(0,1,1),(1-\eta,1;P)\\ (0,1),(-\eta,1;P)\end{matrix}\right.\right]\right.
$$

$$
\left.-e^{-\omega\pi a_{N+1}}\bar{H}_{2,2}^{1,2}\left[-ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(0,1,1),(1-\eta,1;P)\\ (0,1),(-\eta,1;P)\end{matrix}\right.\right]\right\}.
$$

(iii) If $A_j = B_j = 1$ in (2.1) and (2.2) then $\bar{H}$ reduces to Fox H-function and we get known recurrence relations given earlier by Chaurasia [2].

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CONVEX OPTIMIZATION TECHNIQUES DUE TO NESTROV AND COMPUTATIONAL COMPLEXITY

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ABSTRACT

A large body of literature is devoted to the estimation of covariance matrices in a large-scale setting. Recent work in this area includes the shrinkage approach proposed by Schäfer and Strimmer [16], where the authors analystically calculate the optimal shrinkage intensity, yielding a good, computationally inexpensive estimate. Our focus is an estimate with the property that the corresponding inverse covariance matrix is sparse.

Dempster [4] introduced the concept of covariance selection, where the number of parameters to be estimated is reduced by setting to zero some elements of the inverse covariance matix. Covariance selection can lead to a more robust estimate of S if enough entries of its inverse are set to zero. Traditionally, a greedy forward/backward search algorithm is employed to determine the zero patterns Lauritzen [9]. However, this method quickly becomes computationally infeasible as p grows.

2000 Mathematics Subject Classification : Primary 90C25; Secondary 90C47, 49K30

Keywords and Phrases : Applications of Mathematical Optimization, Robustness, Duality and Bounds, Convergence and Property of Solution: Convex Optimization, Nesterov's Method.

1. Introduction. In this paper we investigate the following related idea. Beginning with a dense empirical covariance matrix S, we compute a maximum likelihood estimate of S with an II-norm penalty added to encourage sparsity in the inverse. The authors Li and Gui [10] introduce a gradient descent algorithm in which they account for the sparsity of the inverse covariance matrix by defining a loss function that is the negative of the log likelihood function. Recently, Huang, Liu and Pourahmodi [7], and Dahl [2] considered penalized maximum liklihood estimation, and Dahl [2] in particular, proposed a set of large scale methods to solve problems where a sparse structure of S-1 is known a priori. Our contribution is threefold: we present a provably convergent algorithm that is efficient for large-

scale instances, yielding a sparse, invertible estimate of S-1, even for $n<p$; we obtain some basic complexity estimates for the problem; and finally we test our algorithm on synthetic data as well as gene expression data from two datasets.

Notations. For a $p \times p$ matrix $X, X \geq 0$ means X is symmetric and positive semi-definite; $\|x\|$ denotes the largest singular value norm, $\|x\|_1$ the sum of the absolute values of its elements, and $\|x\|_\infty$ their largest magnitude.

2. Preliminaries. In this section we set up the problem and discuss some of its properties.

2.1. Problem Setup. Let $S \geq 0$ be a given empirical matrix, for data drawn from a multivariate Gaussian distribution. Let the variable $\hat{X}$ be our estimate of the inverse covariance matrix. We consider the penalized maximum-liklihood problem

$$max\, log\, det\, X - (X, X) - \rho \|x\|_1 \qquad ...(1.1)$$

where $\langle S, X \rangle = trace(SX)$ denotes the scalar product between two symmetric matrices S and X, and the term $\|x\|_1 = \sum_{i,j} |X_{ij}|$ penalizes nonzero elements of X.

Here, the scalar parameter $\rho>0$ controls the size of the penalty, hence the sparsity of the solution. The penalty term involving the sum of absolute values of the entries of X is a proxy for the number of its nonzero elements, and is often used-albeit with vector, not matrix, variables-in regression techniques, such as *LASSO* in Tibshirani [17], when sparsity of the solution is a concern.

The classical maximum likelihood estimate of Σ is recovered for $\rho>0$, and is simply S, the empirical covariance matrix. Furthermore, as noted above, for $p>>n$, the matrix S is likely to be singular. It is desirable for our estimate of S to be invertible. We shall show that our proposed estimator performs some regularization, so that our estimate is invertible for every $\rho>0$.

2.2. Robustness, Duality and Bounds. By introducing a dual variable U, we can write (1) as $\max_{X>0} \min_{\|U\|_\infty \leq \rho} log\, det\, X + \langle X, S+U \rangle$.

Here $\|U\|_\infty$ denotes the maximal absolute value of the entries of U. This corresponds to seeking an estimate with maximal worst-case likelihood, over all component wise bounded additive perturbations $S+U$ of the empirical covariance matrix S. Such a "robust optimization" interpretation can be given to a number of estimation problems, most notably support vector machines for classification.

We can obtain the dual problem by exchanging the *max* and the *min*:

$$\min_U \{-log\, det(S+U) - p : \|U\|_\infty \leq \rho, S+U>0\} \qquad (2)$$

The diagonal elements of an optimal U are simply $\hat{U} = \rho$. The

corresponding covariance matrix estimate is $\cdot \sum := S + \hat{U}$. Since the above dual problem has a compact feasible set, the primal and dual problems are equivalent. The optimality conditions relate the primal and dual solutions by $\sum X = 1$.

The following theorem shows that adding the l_1-norm penalty regularizes the solution.

Theorem-1 For every ρ>0, the optimal solution to the penalized *ML* problem (1) is unique, and bounded as follows :

$$\alpha(p)I \le X \le \beta(p)I, \text{ where } \alpha(p) = \frac{1}{\|S\| + \rho p}, \beta(p) = \frac{p}{\rho}.$$

Proof. An optimal X satisfies $X=(S+U)^{-1}$, where $\|U\| \le \rho$. Thus, we can without loss of generality impose that $X \ge \alpha(p)I$, where $\alpha(p)$ is defined in the theorem. Likewise, we can show that X is bounded above. Indeed, at optimum, the primal-dual gap is zero:

$$0 = -log\,det(S+P) - p - log\,det\,X + \langle S, X\rangle + \rho\|X\|_1$$
$$= -p + \langle S, X\rangle + \rho\|X\|_1,$$

where we have used $(S+U)X=1$. Since S,X are both positive semi-definite, we obtain $\|X\| \le \|X\|_F \le \|X\|_1 \le \beta(p)I$ as claimed. Problem (2) is smooth and convex. When $p(p+1)/2$ is in the low hundreds, the problem can be solved by existing software that uses an interior point method Vandenberghe [18], The complexity to compute an ε-suboptimal solution using such-second-order methods, however, is $O(p^6 log(1/\varepsilon))$ making them infeasible for even moderately large p.

The authors Dahl et al. [2] developed a set of algorithms to estimate the nonzero entries of Σ^{-1} when the sparsity pattern is known a priori and corresponds to an undirected graphical model that is not chordal. Here our focus is on relatively large, dense problems, for which the sparsity pattern is not known a priori. Note that we cannot expect to do better than $O(p^3)$, which is the cost of solving the non-penalized problem ρ=0 for a dense sample covariance matrix S.

2.3 Choice of Regularization Parameter ρ. In this section we provide a simple heuristic for choosing the penality parameter ρ, based on hypothesis testing, We emphasize that while the choice of ρ is an important issue that deserves a thorough investigation, It is not the focus of this paper.

The heuristic is based on the observation that if $\rho < |S_{ij}|$ then there cannot be zero

in that element of our estimate of the covariance matrix $\Sigma_{ij} \neq 0$ suppose we choose ρ according to

$$\rho = \frac{t_{n-2}(\gamma) max_{i,j} S_{ii} S_{ji}}{\sqrt{n-2+t_{n-2}^2(\gamma)}}, \tag{3}$$

where $t_{n-2}(\gamma)$ denotes the two-tailed 100γ% point of the t-distribution, for n-2 degrees of freedom. With this choice, and using the fact that $S \geq 0$, it can be shown that $\rho < |S_{ij}|$ implies the condition for rejecting the null hypothesis that variables i and j are independent in the underlying distribution, under a likelihood ratio test of size γ Muirhead [13]. We note that this choice yields an asymtotically consistent estimator. As $n \to \infty$, we recover the sample covariance S as our estimate of the covariance matrix, and S converges to the true covariance Σ.

3. Block Coordinate Descent Methd. In this section we present an efficient algorithm for solving the dual problem (2) based on block coordinate descent.

3.1 Algorithm. We first describe a method for solving (2) by optimizing over one column and row of $S+U$ at a time. Let $W:=S+U$ be our estimate of the true covariance. The algorithm begins by initializing $W^0=S+\rho I$. The diagonal elements of W^0 are set to their optimal values, and are left unchanged in what follows.

We can permute rows and columns of W, so that we are optimizing over the last column and row. Partition W and S as

$$W = \begin{pmatrix} W_{11} & w_{12} \\ w_{12}^T & w_{22} \end{pmatrix} \qquad S = \begin{pmatrix} S_{11} & s_{12} \\ s_{12}^T & s_{22} \end{pmatrix}$$

where $w_{12}, s_{12} \in R^{p-1}$ the update rule is found by solving the dual problem (2), with U fixed except for its last column and row. This leads to a box-constrained quadratic program (QP):

$$\hat{w}_{12} = arg\,min\left\{ y^T W_{11}^{-1} y : \|y - s_{12}\|_\infty \leq \rho \right\} \tag{4}$$

We cycle through the columns in order, solving a QP at each step. After each sweep through all columns, we check to see if the primal-dual gap is less than ε, a given tolerance. The primal variable is related to W by $X=W^{-1}$. The duality gap condition is then

$$\langle S, X \rangle + \rho \|X\|_1 \leq p + \varepsilon.$$

3.2. Convergence and Property of Solution. Iterates produced by the coordinate descent algorithm are strictly positive definite. Indeed, since $S \geq 0$, we have that $W^0 > 0$ for any $\rho > 0$. Now suppose that at iteration ...

that the following Schur complement is positive : $w_{22} - w_{12}^T W_{11}^{-1} w_{12} > 0$. By the update rule (4), we have

$$w_{22} - \hat{w}_{12} W_{11}^{-1} \hat{w}_{12} > w_{22} - w_{12}^T W_{11}^{-1} w_{12} > 0 \ ,$$

which, using Schur complements again, implies that the new iterate satisfies $\hat{W} > 0$. Note that since the method generates a sequence of feasible primal and dual points, the stopping criterion is nonheuristic. As a consequence, the QP (4) to be solved at each iteration has a unique solution. This implies that the method converges to the true solution of (2), by virtue of general results on block-coordinate descent algorithms Bertsekas [1].

The above results shed some interesting light on the solution to problem (2). Suppose that the column s_{12} of the sample covariance satisfies $|s_{12}| \le \rho$, where the inequalities hold component wise. Then the corresponding column of the solution is zero $\Sigma_{12} = 0$. Indeed, if the zero vector is in the constraint set of the QP (4), then it must be the solution to that QP. As the constraint set will not change no matter how many times we return to that column, the corresponding column of all iterates will be zero. Since the iterates converge to the solution, the solution must have zero for that column. This property can be used to reduce the size of the problem in advance, by setting to zero columns of W that correspond to columns in the sample covariance S that meet the above condition.

Using the work of Luo and Tseng [11], it is possible to show that the local convergence rate of this method is at least linear. In practice we have found that a small number of sweeps through all columns, independent of problem size p, is sufficient to achieve convergence. For a fixed number of K sweeps, the cost of the method is $O(Kp^4)$, since each iteration costs $O(p^3)$.

3.3 Connection to *LASSO*. The dual of (4) is

$$\min_x x^T W_{11} x - s_{12}^T + \rho \|x\|_1 \ . \tag{5}$$

Strong duality obtains so that problems (5) and (4) are equivalent. If we let Q denote the square root of $W_{11}, b := \frac{1}{2} Q^{-1}$, then we can write(5) as $\min_x \|Qx - b\|_2^2 + \rho\|x\|_1$

The above is a penalized least-squares problem, often referred to as *LASSO*. If W_{11} were a principal minor of the sample covariance S, then the above would be equivalent to a penalized regression of one variable against all others. Thus, the approach is reminiscent of the approach explored by Meinshausen and Buhlmann [12]. but there are two major differences. First, we begin with some regularization, and as a consequence, each penalized regression problem has a unique solution. Second, and more importantly, we update the problem data after each regression;

in particular, W_{11} is never a minor of S. In a sense, the coordinate descent method can be interpreted as a recursive *LASSO* method.

4. Nesterov's Method. In this section, we apply the recent results due to Nesterov [15] to obtain a first-order method for solving (1). Our main goal is not to obtain another algorithm, as we have found that the coordinate descent is already quite efficient; rather, we seek to use Nesterov's formalism to derive a rigorous complexity estimate for the problem, improved over that delivered by interior point methods.

As we shall see, Nesterov's framework allows us to obtain an algorithm that has a complexity of $O\left(p^{4.5}/\varepsilon\right)$, where $\varepsilon>0$ is the desired accuracy on the objective of problem (1). This is to be contrasted with the complexity of interior-point methods, $O\left(p^{6}\,log\left(1/\varepsilon\right)\right)$. Thus, Nesterov's method provides a much better dependence on problem size, at the expense of a degraded dependence on accuracy. In our opinion, obtaining an estimate that is accurate numerically up to dozens of digits has little practical value, as it is much more important to be able to solve larger problems with less accuracy. Note also that the memory requirements for Nesterov's methods are much better than those of interior-point methods.

4.1 Idea of Nesterov's Method. Nesterov's method [15] applies to a class of non-smooth, convex optimization problems, of the form

$$\min_{x}\left\{f(x): x \in Q_1\right\} \tag{6}$$

where the objective function is described as

$$f(x)=\hat{f}(x)+\max_{u}\left\{\langle Ax,u\rangle_2 : u \in Q_2\right\}.$$

Here Q_1 and Q_2 are bounded, closed, convex sets, $\hat{f}(x)$ is differentiable (with Lipschitz continuous gradient) and convex on Q_1, and A is a linear operator. Observe that we can write (1) in this form if we impose bounds on the eigenvalues of the solution, X. To this end, we let

$$Q_1=\left\{X : \alpha I \le X \le \beta I\right\},$$

$$Q_2=\left\{U : \|U\|_\infty \le \rho\right\},$$

where $\alpha,\beta\,(0<\alpha<\beta)$ are given. We also define

$$\hat{f}(X)=-log\,det\,X+\langle S,X\rangle, \text{ and } A=\rho I.$$

To Q_1 and Q_2, we associate norms and continuous, strongly convex functions, called prox-functions, $d_1(X)$ and $d_2(U)$. For Q_1 we choose the Frobenius norm, an a prox-function $d_1(X)=-log\,det\,X+log\,\beta$. For Q_2, we choose the Frobenius norm again, and a prox-function $d_2(U)=\|U\|_F^2/2$.

The method applies a smoothing technique to the non-smooth problem (6), which replaces the objective of the original problem, $f(X)$, by a penalized function involving the prox-function $d_2(U)$:

$$\tilde{f}(X) = \hat{f}(X) + \underset{U \in Q_2}{mix}\{\langle AX, U\rangle - \mu d_2(U)\}. \tag{7}$$

The above function turns out to be a smooth uniform approximation. It is differentiable, convex on Q_1, and has a Lipschitz-continuous gradient, with a constant L that can be computed as detailed below. A specific gradient scheme is then applied to this smooth approximation, with convergence rate $O(L/\varepsilon)$.

4.2 Algorithm and Complexity Estimate. To detail the algorithm and compute the complexity, we first calculate some parameters corresponding to our definitions above. First, the strong convexity parameter for $d_1(X)$, on Q_1 is $\sigma_1 = 1/\beta^2$, in the sense that $\nabla^2 d_1(X)[H,H] = trace\left(X^{-1}HX^{-1}H\right) \geq \beta^{-2}\|H\|_F^2$ for every symmetric H. Furthermore, the center of the set Q_1 is $X_0 = arg\,min_{X \in Q_1} d_1(X) = \beta I$, and satisfies $d_1(X_0) = 0$. Without choice, we have $D_1 = max_{x \in Q_1} d_1(X) = \rho\, log\, \beta/\alpha$.

Similary, the strong convexity parameter for $d_2(U)$ on Q_2 is $\sigma_2 = 1$ and we have $D_2 = max_{U \in Q_2} d_2(U) = p^2/2$. With this choice, the center of the set Q_2 is $U_0 = arg\,min_{U \in Q_2} d_2(U) = 0$.

For a desired accuracy ε, we set the smoothness parameter $\mu = \varepsilon/2D_2$, and start with the initial point $X_0 = \beta I$. The algorithm proceeds as follows.

For $k \geq 0$ do

1. Compute $\nabla\tilde{f}(X_k) = -X_k^{-1} + S + U^*(X_k)$ where $U^*(X)$ solves (7).

2. Find $Y_k = arg\,min_Y\left\{\left\langle \nabla\tilde{f}(X_k), Y - X_k\right\rangle + \frac{1}{2}L(\varepsilon)\|Y - X_k\|_F^2 : X \in Q_1\right\}$.

3. Find $Z_k = arg\,min_X\left\{\frac{L(\varepsilon)}{\sigma_1}d_1(X) + \sum_{i=0}^{k}\frac{i+1}{2}\left\langle \nabla\tilde{f}(X_i), X - X_i\right\rangle : X \in Q_1\right\}$.

4. Update $X_k = \dfrac{2}{k+3}Z_k + \dfrac{k+1}{k+3}Y_k$.

In our case, the Lipschitz constant for the gradient of our smooth approximation to the objective function is $L(\varepsilon) = M + D_2\|A\|^2/(2\sigma_2\varepsilon)$, where

$M = 1/\alpha^2$ is the Lipschitz constant for the gradient of $\tilde{f}$, and the norm $\|A\|$ is induced by the Frobenius norm, and is equal to ρ. The algorithm is guaranteed to produce an ε-suboptimal solution after a number of steps not exceeding

$$N(\varepsilon) = 4\|A\|\sqrt{\frac{D_1 D_2}{\sigma_1 \sigma_2}\frac{1}{\varepsilon}} + \sqrt{\frac{MD_1}{\sigma_1 \varepsilon}} \tag{8}$$

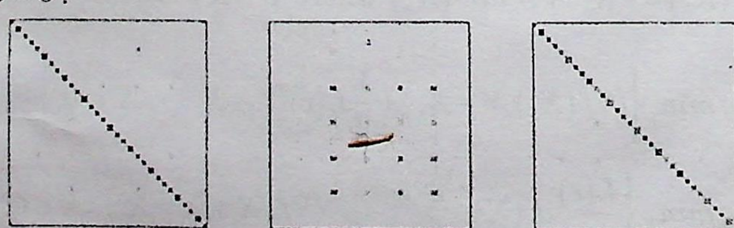
$$= \frac{k\sqrt{p(\log k)}}{\varepsilon}\left(4p\alpha + \sqrt{\varepsilon}\right),$$

where $k = \beta/\alpha$ is bound on the condition number of the solution.

Now we are ready to estimate the complexity of the algorithm. For step 1, the gradient of the smooth approximation is readily computed in closed form, via the computation of the inverse of X. Step 2 essentially amounts to projecting on Q_1, and requires an eigen value problem to be solved; likewise for step 3. In fact, each iteration costs $O(p^3)$. The number of iterations necessary to achieve an objective with absolute accuracy less than ε is given in (8) by, $N(\varepsilon) = O(p^{1.5}/\varepsilon)$, if the condition number k is fixed a priori. Thus, the Complexity of the algorithm is $O(p^{4.5}/\varepsilon)$.

5. Numerical Results. In this section we present some numerical results. We begin with a small synthetic example to test the ability of the method to recover a sparse structure from a noisy matrix. Starting with a sparse matrix A, we obtain S by adding a uniform noise of magnitude $\sigma = 0.1$ to A^{-1}.

In figure 1 we plot the sparsity patterns of A, S^{-1}, and the solution $\hat{X}$ to (1) using S and $\rho = \sigma$.



A $\qquad$ S^{-1} $\qquad$ $\hat{X}$

Figure 1. Recovering the Sparsity Pattern

We plot the underlying sparse matrix A, the inverse of the noisy version of A^{-1}, and the solution to problem (1) for ρ equal to the noise level.

We next perform the following experiment to see what happens to the solution of (1) as we vary the parameter ρ above and below the noise level σ. For each value of

ρ, we randomly generate 10 sparse matrices A of size $n=50$. We then obtain sample covariance matrices S as above, again using $\sigma=0.1$.

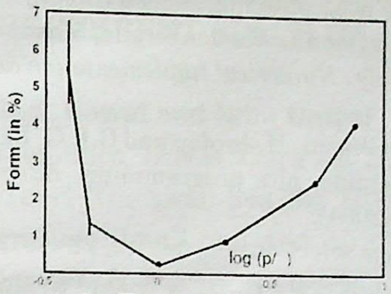


Figure 2. Recovering Structure

Gene Expression Properties

The 300 experiment compendium dataset contains $n=253$ samples with $p=6136$ variables. With a view towards obtaining a very sparse graph, we set $\gamma=0.1$ in the heuristic formula (3) of section (2.3) to obtain $\rho=0.0313$.

Applying the property of the solution discussed in section (3.2), the size of the problem was reduced to $\hat{p}=537$. Three sweeps through all columns were required to achieve a duality gap of $\varepsilon = 0.146$, with a total computing time of 18-minutes, 34-seconds. The resulting estimate of the inverse covariance matrix $\hat{\Sigma}^{-1}$ is 99% sparse and has a condition number of 21.84. Figure (4) shows a sample subgraph obtained from $\hat{\Sigma}^{-1}$, generated using the Graph Explore program developed by Dobra and West [6]. The method has picked out a cluster of genes associated with amino acidmetabolism, as described by Hughes et al. [7].

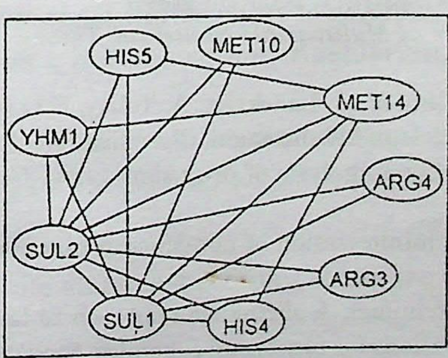


Figure 3. Application to Hughes Dataset Using ρ=0.0313.

As we have seen, the penalized maximum likelihood problem formulated here is useful for recovering a sparse underlying precision matrix Σ^{-1} from a dense sample covariance matrix S, even when the number of samples n is small relative to the number of variables p. In preliminary tests, the method appears to be a potentially valuable tool for analyzing gene expression data, although further testing is

required.

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ON DECOMPOSITION OF CURVATURE TENSOR FIELDS IN A KAEHLERIAN RECURRENT SPACE OF FIRST ORDER

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ABSTRACT

Takano [13] studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [12] defined and studied defined decomposition of recurrent curvature tensor field in a Finsler space. Negi and Rawat ([2][3],[4]) studied decomposition of recurrent curvature tensor field in a Kaehlerian space. Rawat [5], Rawat and Silswal ([6],[7]) defined and studied decomposition of recurrent curvature tensor fields in a Tachibana space. Further, Rawat and Dobhal ([8],[9]) studied decomposition of recurrent corvature tensor field in a Kaehlerian recurrent space. Rawat and Singh ([10],[11]) studied the decomposition of curvature tensor fields in a Kaehlerian recurent space of first order.

In the present paper, we consider the decomposition of curvature tensor field Rh^i_{jk} in terms of two non-zero vectors and a tensor field. Also several theorems are established and proved therein.

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1. Introduction. When in a $2n$-dimensional real space X_{2n} of class $C^r(r \geq 2)$, there is a mixed tensor field $F_i^h; R_{i,j} = 1,2,3,...,2n$ satisfying

$$F_i^h F_l^h = -A_j^h, \qquad ...(1.1)$$

we say that the space admits an almost complex structure and we call such a space an almost complex space.

If an almost complex space has a positive definite Riemannian metric $ds^2 = g_{ji} d\xi^j d\xi^i$ which setisfies

$$F_j^l F_i^k g_{lk} = g_{ji}, \qquad ...(1.2)$$

Then the space is called an almost-Hermitian space.

In this case the tensor $F_{..} \overset{def}{=} F_i^l g_{lh}$ is anti-symmetric(or skew-symmetric) in i and h.

If an almost-Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \quad \text{...(1.3)}$$

where ∇_j denotes the operator of covariant differentiation with respect to the metric tensor g_{ji} of the Riemannian space then it is called an almost-Kaehlerian space and if it satisties

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0, \quad \text{...(1.4)}$$

then it is called a K-space.

In an Almost-Hermitian space, if

$$\nabla_j F_{ih} = 0, \text{ or } F_{ih,j} = 0, \quad \text{...(1.5)}$$

then it is called a Kaehlerian space.

The Riemannian curvature tensor field is defined by

$$R^h_{ijk} = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\}\left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\}\left\{ {l \atop ik} \right\}, \quad \text{...(1.6)}$$

where $\partial_i = \partial / \partial x^i$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the scalar curvature are given by

$R_{ij} = R^a_{aij}$ and $R = R_{ij} g^{ij}$ respectively.

It is well known that these tensors satisfy the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \quad \text{...(1.7)}$$

$$R_{,i} = 2R^a_{i,a} \quad \text{...(1.8)}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \quad \text{...(1.9)}$$

and $\quad F^a_i R^j_a = R^a_i F^j_a. \quad \text{...(1.10)}$

The holomorphically projective curvature tensor is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right), \quad \text{...(1.11)}$$

where $S_{ij} = F^a_i R_{aj}$.

The Bianchi identities in K_n are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \quad \text{...(1.12)}$$

and $\quad R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0. \quad \text{...(1.13)}$

The commutative formulae for the curvature tensor field are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} \quad \text{...(1.14)}$$

and $\quad T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}.$ ...(1.15)

A Kaehlerian space K_n is said to be Kaehlerian recurrent space of first order, if its curvature

$$\nabla_a R^h_{ijk} = \lambda_a R^h_{ijk},$$

i.e. $\quad R^h_{ijk,a} = \lambda_a R^h_{ijk},$ ...(1.16)

where λ_a is a non-zero vector and is known as recurrent vector field. The space is said to be Ricci-recurrent space of first order, if it satisfies the condition

$$R_{ij,a} = \lambda_a R_{ij}, \qquad \text{...(1.17)}$$

Multiplying the above equation by g^{ij}, we have

$$R_{,a} = \lambda_a R. \qquad \text{...(1.18)}$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}. We consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = v'^h \phi_i \psi_{j,k} \qquad \text{...(2.1)}$$

where two vectors u'^h, ϕ_i and tensor field $\psi_{j,k}$ are such that

$$v'^h \lambda_h = 1. \qquad \text{...(2.2)}$$

Theorem 2.1. Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} take the forms

$$\phi_i \psi_{j,k} + \phi_j \psi_{k,i} + \phi_k \psi_{i,j} = 0 \qquad \text{...(2.3)}$$

and $\quad \lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j} = 0.$ ...(2.4)

Proof. From equations (1.12) and (2.1), we have

$$\phi_i \psi_{j,k} + \phi_j \psi_{k,i} + \phi_k \psi_{i,j} = 0. \qquad \text{...(2.5)}$$

Since $\qquad v'^h \neq 0 \cdot$

From equations (1.13), (1.16) and (2.1), we get

$$v'^h \phi_i \left[\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j}\right] = 0. \qquad \text{...(2.6)}$$

Multiplying (2.6) by λ_h and using (2.2), we obtain

$$\phi_i \left[\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j}\right] = 0. \qquad \text{...(2.7)}$$

Since $\phi_i \neq 0$, therefore, we derive

$$\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j} = 0.$$

This completes the proof of the theorem.

Theorem 2.2. Under the decomposition (2.1), The tensor field R^h_{ijk}, R_{ij} and $\psi_{j,k}$ satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = \phi_i \psi_{j,k} \qquad ...(2.8)$$

Proof. With the help of equations (1.7), (1.16) and (1.17), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \qquad ...(2.9)$$

Multiplying (2.1) by λ_h and using relation (2.2), we get

$$\lambda_h R^h_{ijk} = \phi_i \psi_{j,k} \qquad ...(2.10)$$

From equations (2.9) and (2.10, we derive the required relation (2.8).

Theorem 2.3. Under the decomposition (2.1), the quantities λ_a and υ'^h behave the recurrent vectors. The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a \qquad ...(2.11)$$

and $$\upsilon'^h_{,m} = -\mu_m \upsilon'^h \qquad ...(2.12)$$

Proof. Differentiating (2.8) covariantly w.r.t. x^m and using (2.1) and (2.8), we obtain

$$\lambda_{a,m} \upsilon'^h \phi_i \psi_{j,k} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \qquad ...(2.13)$$

Multiplying (2.13) by λ_a and using (2.1) and (2.9), we have

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right) = \lambda_a \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right), \qquad ...(2.14)$$

Now, multiplying equation (2.14) by λ_h we get

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right)\lambda_h = \lambda_a \lambda_h \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right). \qquad ...(2.15)$$

Since the expression on the right hand side of above equation is symmetric in a and h, therefore

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a, \qquad ...(2.16)$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

The vector field λ_a being non-zero, we can have a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a \qquad ...(2.17)$$

Further, differentiating the equation (2.2) w.r.t. x^m, we have

$$\lambda_h \upsilon'^h_{,m} + \lambda_{h,m} \upsilon'^h = 0.$$

Making use of equation (2.11), we derive

$$v'^h_{,m} = -\mu_m v'^h \quad (\text{since } \lambda_h \neq 0). \qquad \text{...(2.18)}$$

This proves the theorem.

Theorem 2.4 Under the decomposition (2.1), the vector ϕ_i and the tensor $\psi_{j,k}$ satisfy the equation

$$\phi_i \psi_{j,k}(\lambda_m + \mu_m) = \phi_i \psi_{j,km} + \psi_{j,k}\phi_{i,m}, \qquad \text{...(2.19)}$$

where $\psi_{j,km} = \psi_{j,k,m}$.

Proof. Differentiating (2.1) covariantly w.r.t. x^m and using equations (1.16), (2.1) and (2.12), we get the required result of the Theorem.

Theorem 2.5 Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal if

$$\psi_{k,m}\left\{(\phi_i\delta^h_j - \phi_j\delta^h_i) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2\phi_l\psi_{j,m}F^h_k F^l_i = 0. \qquad \text{...(2.20)}$$

Proof. The equation (1.11) may be written in the form

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk}, \qquad \text{...(2.21)}$$

where

$$D^h_{ijk} = \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right). \qquad \text{...(2.22)}$$

Contracting indices h and k in (2.1), we have

$$R_{ij} = v'^k \phi_i \psi_{j,k}. \qquad \text{...(2.23)}$$

In view of equation (2.23), we get

$$S_{ij} = F^l_i v'^m \phi_l \psi_{j,m}. \qquad \text{...(2.24)}$$

Making use of relations (2.23) and (2.24) in equation (2.22), we have

$$D^h_{ijk} = \frac{1}{(n+2)}\left[v'^m \psi_{k,m}\left\{(\phi_i\delta^h_j - \phi_j\delta^h_i) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2v'^m \phi_l\psi_{j,m}F^h_k F^l_i\right] = 0 \qquad \text{...(2.25)}$$

From (2.21), it is clear that $P^h_{ijk} = R^h_{ijk}$ if $D^h_{ijk} = 0$,

which, in view of (2.25), becomes

$$v'^m \psi_{k,m}\left\{(\phi_l\delta^h_j - \phi_j\delta^h_l) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2v'^m \phi_l\psi_{j,m}F^h_k F^l_i = 0 \qquad \text{...(2.26)}$$

Multiplying the above equation by λ_m and using relation (2.2), we obtain the required result (2.20).

Theorem 2.6 Under the decomposition (2.1), the scalar curvature R, satisfies the

$$\lambda_k R = R_{,k} = g^{ij}\phi_i \psi_{j,k}.$$

Proof. Contracting indices h and k in (2.1), we have

$$R_{ij} = v^{ik}\phi_i \psi_{j,k}. \qquad \text{...(2.27)}$$

Multiplying (2.27) by g^{ij} both sides, we have

$$R = g^{ij} v^{ik}\phi_i \psi_{j,k}. \qquad \text{...(2.28)}$$

Multiplying (2.28) by λ_k and using (2.2), we get

$$\lambda_k R = g^{ij}\phi_i \psi_{j,k}$$

or, $\quad R_{,k} = g^{ij}\phi_i \psi_{j,k}$ [by using (1.18)] .

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FOURIER SERIES FOR MODIFIED MULTI-VARIABLE *H*-FUNCTION

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ABSTRACT

The object of this paper is to derive integral involving modified H-function of several variables. These integrals are used to establish the Fourier series for generlized function. By suitably specializing the coefficients and the parameters we can obtain many (new and known) interesting results involving multi-variable H-functions [8]. The results obtained by Srivastava and Panda [8], Kaul [1,2], Mac Robert [3] and Sneddon [7] follow as particular cases of our results.

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1. Introduction. The modified multi-variable H-function employed as kernel of multi-dimentional transform defined by Prasad and Singh [6] on the lines of Srivastava and Panda [8], Prasad and Maurya [5] is as follows:

$$H^{m,n:|R':m_1,n_1;\dots;m_r,n_r}_{p,q:|R:p_1q_1;\dots;p_r,q_r}\left[\begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix}\left| \begin{matrix} \left(a_j;\alpha'_j,\dots,\alpha^{(r)}_j\right)_{1,p} : \left(e_j : u'_j g'_j,\dots,u^{(r)}_j g^{(r)}_j\right)_{1,IR'} : \left(c'_j,\gamma'_j\right)_{1,p_1};\dots;\left(c^{(r)}_j\gamma^{(r)}_j\right)_{1,p_r} \\ \left(b_j : \beta'_j,\dots,\beta^{(r)}_j\right)_{1,q} : \left(l_j;U'_j f'_j,\dots,U^{(r)}_j f^{(r)}_j\right)_{1,IR} : \left(d'_j,\delta'_j\right)_{1,q_1};\dots;\left(d^{(r)}_j\delta^{(r)}_j\right)_{1,q_r} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\times\psi(\xi_1,\dots,\xi_r)\,z_1^{\xi_1}\dots z_r^{\xi_r}\,d\xi_1\dots d\xi_r \qquad \dots(1.1)$$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i}\Gamma\left(d^{(i)}_j\xi_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-c^{(i)}_j-\gamma^{(i)}_j\xi_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d^{(i)}_j-\delta^{(i)}_j\xi_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(c^{(i)}_j-\gamma^{(i)}_j\xi_i\right)} \qquad (i=1,2,\dots,r) \qquad \dots(1.2)$$

$$\psi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j^{(i)}+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}\times\frac{\prod_{j=1}^{IR'}\Gamma\left(e_j+\sum_{i=1}^{r}u_j^{(i)}g_j^{(i)}\xi_i\right)}{\prod_{j=1}^{IR'}\Gamma\left(I_j^{(i)}+\sum_{i=1}^{r}U_j^{(i)}f_j^{(i)}\xi_i\right)}.$$

...(1.3)

The multiple integral (1.1) converges absolutely if

$$|arg\, z_i|<\frac{1}{2}U_i\pi, \qquad (i=1,2,\ldots,r).$$

where

$$U_i=\sum_{j=1}^{m}\beta_j^{(i)}-\sum_{j=m+1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n}\alpha_j^{(i)}-\sum_{j=n+1}^{p}\alpha_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}$$

$$-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{IR'}g_j^{(i)}-\sum_{j=1}^{IR}f_j^{(i)}>0 \qquad (i=1,2,\ldots,r).$$

We shall require the following known integrals.

1. $\int_0^{\pi}\cos P\theta(\cos\theta/2)^{2\rho}(\sin\theta/2)^{2\rho_1}\,d\theta$

$$=\frac{\Gamma(P+\rho+1/2)\Gamma(\rho_1+1/2)}{2\Gamma(P+\rho+\rho_1+1)}\,{}_3F_2\left[\begin{matrix}\rho_1+1/2,-P,-P+1/2;\\-P-\rho+1/2,1/2;\end{matrix}1\right] \quad ...(1.4)$$

provided that $Re(2\rho+1)>0$, $Re(\rho_1+1)>0$ and $P=0,1,2,\ldots$.

2. $\int_0^{\pi}\sin(2s+1)\theta(\cos\theta)^{2\rho}(\sin\theta)^{2\rho_1}\,d\theta$

$$=\frac{\Gamma(2s+2)\Gamma(s+\rho+1/2)\Gamma(\rho_1+1)}{\Gamma(s+\rho+\rho_1+3/2)\Gamma(2s+1)}\,{}_3F_2\left[\begin{matrix}\rho_1+1,-s,(-s+1/2);\\(-s-\rho+1/2),3/2;\end{matrix}1\right],$$

provided that $Re(2\rho+1)>0$, $Re(\rho_1+1)>0$ and $s=0,1,2,\ldots$.

2. Main Integrals.

(I) $$\int_0^{\pi}\left(\cos\frac{\theta}{2}\right)^{2\rho}\left(\sin\frac{\theta}{2}\right)^{2\rho_1}H\left[\left\{\left(\cos\frac{\theta}{2}\right)^{2h_1}\left(\sin\frac{\theta}{2}\right)^{2k_1}z_1,\ldots,\left(\cos\frac{\theta}{2}\right)^{2h_r}\left(\sin\frac{\theta}{2}\right)^{2k_r}z_r\right\}\right]d\theta$$

$$= H^{m,n+2:IR,m_1,n_1:\ldots;m_r,n_r}_{p+2,q+1:IR',p_1,q_1:\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} (1/2-\rho;h_1,\ldots,h_r),(1/2-\rho_1;k_1,\ldots,k_r), \\ (-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j\right)_{1,p}:\left(e_j:u'_jg'_j,\ldots,u^{(r)}_jg^{(r)}_j\right)_{1,IR'}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c^{(r)}_j,\gamma^{(r)}_j\right)_{1,p_r} \\ \left(b_j:\beta'_j,\ldots,\beta^{(r)}_j\right)_{1,q}:\left(l_j;U'_jf'_j,\ldots,U^{(r)}_jf^{(r)}_j\right)_{1,IR}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d^{(r)}_j\delta^{(r)}_j\right)_{1,q_r} \end{matrix}\right] \quad \ldots(2.1)$$

provided that $Re\left(2\rho+2\sum_{i=1}^{r}h_i\lambda_i\right)>0$, $Re\left(2\rho_1+2\sum_{i=1}^{r}k_i\lambda_i+1\right)>0$,

where $\lambda_i = min\,Re\left(d^{(i)}_j/\delta^{(i)}_j\right)\ (j=1,\ldots m_i,i=1,\ldots,r)$

(iii) $0\le\theta\le\pi$

along with the conditions for the convergence of modified H-function.

$$\text{(II)}\int_0^\pi cos\,P\theta\left(cos\frac{\theta}{2}\right)^{2\rho}\left(sin\frac{\theta}{2}\right)^{2\rho_1}H\left[\left\{\left(cos\frac{\theta}{2}\right)^{2h_1}\left(sin\frac{\theta}{2}\right)^{2k_1}z_1,\ldots,\left(cos\frac{\theta}{2}\right)^{2h_r}\left(sin\frac{\theta}{2}\right)^{2k_r}z_r\right\}\right]d\theta$$

$$=\sum_{N=0}^{P}\frac{(-1)^N(-P)_N(-P+1/2)_N}{(1/2)_N\,N!}H^{m,n+2:IR':m_1,n_1,\ldots,m_r,n_r}_{p+2,q+1:IR:p_1,q_1,\ldots,p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} (1/2+N-P-\rho;h_1,\ldots,h_r),(1/2-N-\rho_1;k_1,\ldots,k_r),\left(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j\right)_{1,p}: \\ (-P-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r),\qquad\qquad \left(b_j:\beta'_j,\ldots,\beta^{(r)}_j\right)_{1,q}: \end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(e_j:u'_jg'_j,\ldots,u^{(r)}_jg^{(r)}_j\right)_{1,IR'}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c^{(r)}_j,\gamma^{(r)}_j\right)_{1,p_r} \\ \left(l_j;U'_jf'_j,\ldots,U^{(r)}_jf^{(r)}_j\right)_{1,IR}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d^{(r)}_j,\delta^{(r)}_j\right)_{1,q_r} \end{matrix}\right] \quad \ldots(2.2)$$

with the conditions of (2.1) and $P=0,1,2,\ldots$.

$$\text{(III)}\int_0^\pi sin(2P+1)\theta(cos\theta)^{2\rho}(sin\theta)^{2\rho_1}H\left[\left\{(cos\theta)^{2h_1}(sin\theta)^{2k_1}z_1,\ldots,(cos\theta)^{2h_r}(sin\theta)^{2k_r}z_r\right\}\right]d\theta$$

$$=(2P+1)\sum^{P}\frac{(-1)^N(-P)_N(-P+1/2)N}{(3/2)_N\,N!}H^{m,n+2:IR':m_1,n_1,\ldots,m_r,n_r}_{p+2,q+1:IR:p_1,q_1,\ldots,p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} (1/2+N-P-\rho;h_1,\ldots,h_r),(-N-\rho_1;k_1,\ldots,k_r)\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)}\right)_{1,p}: \\ (-1/2-P-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r) \qquad \left(b_j:\beta'_j,\ldots,\beta_j^{(r)}\right)_{1,q}: \end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(e_j:u'_jg'_j,\ldots,u_j^{(r)}g_j^{(r)}\right)_{1,IR'}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c_j^{(r)}\gamma_j^{(r)}\right)_{1,p_r} \\ \left(l_j;U'_jf'_j,\ldots,U_j^{(r)}f_j^{(r)}\right)_{1,IR}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d_j^{(r)}\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right] \qquad \ldots(2.3)$$

provided that $Re\left(\rho_1+\sum_{i=1}^{r}k_i\alpha_i+1\right)>0$,

where $\alpha_i = min\,Re\left(d_j^{(i)}/\delta_j^{(i)}\right),(j=1,2,\ldots,m_i,i=1,2,\ldots,r)$ and $P=0,1,2\ldots$

3. Proof. In order to prove (2.1), we substitute for the modified multi-variable H-function in terms of its contour integral of Mellin-Barenes type and change the order of integration (which is permissible). We then evaluate the inner integral using Gamma function. On interpreting the resulting contour integral by means of (1.1), we obtain (2.1). Proceeding in the similar way, using (1.4) and (1.5) we obtain (2.2) and (2.3).

4. Fourier Series. We shall now establish following Fourier series for mdified multi-variable H-functions :

$$(cos\theta/2)^{2\rho}(sin\theta/2)^{2\rho_1}H\left[\left\{(cos\theta/2)^{2h_1}(sin\theta/2)^{2k_1}z_1,\ldots,(cos\theta/2)^{2h_r}(sin\theta/2)^{2k_r}z_r\right\}\right]$$

$$=\frac{1}{\pi}H^{m,n+1:IR':m_1,n_1;\ldots;m_r,n_r}_{p+1,q+1:IR:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} \left(\frac{1}{2}-\rho;h_1,\ldots,h_r\right), \qquad \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)}\right)_{1,p}: \\ (-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r),\left(b_j;\beta'_j,\ldots,\beta_j^{(r)}\right)_{1,q}: \end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(e_j:u'_jg'_j,\ldots,u_j^{(r)}g_j^{(r)}\right)_{1,IR}:\left(c'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(l_j:U'_jf'_j,\ldots,U_j^{(r)}f_j^{(r)}\right)_{1,IR}:\left(d'_j,\delta'_j\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right]+\frac{2}{\pi}\sum_{P=1}^{\infty}\left\{\sum_{s=0}^{P}\frac{(-P)_s(-P+1/2)_s}{(1/2)_s\,s!}\right\}$$

$$H^{m,n+2:IR':m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:IR:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} (1/2+s-P-\rho;h_1,\ldots,h_r),(1/2-s-\rho_1;k_1,\ldots,k_r): \\ (-P-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r): \end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(a_j;\alpha'_j,\dots,\alpha_j^{(r)}\right)_{1,p} : \left(e_j : u'_j g'_j,\dots,u_j^{(r)} g_j^{(r)}\right)_{1,IR} : \left(c'_j,\gamma'_j\right)_{1,p_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta'_j,\dots,\beta_j^{(r)}\right)_{1,q} : \left(\ell_j : U'_j f'_j,\dots,U_j^{(r)} f_j^{(r)}\right)_{1,IR} : \left(d'_j,\delta'_j\right)_{1,q_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right] \quad \dots(4.1)$$

with the conditions as in (2.1) and $P=0,1,2,\dots$

5. Proof. In order to prove (4.1), let us suppose

$$f(\theta)=(cos\theta/2)^{2\rho}(sin\theta/2)^{2\rho_1} \quad \dots(5.1)$$

$$H\left[\left\{(cos\theta/2)^{2h_1}(sin\theta/2)^{2k_1}z_1,\dots,(cos\theta/2)^{2h_r}(sin\theta/2)^{2k_r}z_r\right\}\right]$$

$$=\frac{1}{2}C_0+\sum_{P=1}^{\infty}C_p\,cos\,P\theta\,. \quad \dots(5.2)$$

Integrating (5.1) between the limits 0 to π and using the reult (2.3), we obtain the value of C_0. Again multiplying both the sides of (5.1) by $cos\,P\theta$ and integrating from 0 to π w.r.t. θ, using (2.2) we get

$$C_p=\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{(-1)_s(-P)_s(-P+1/2)_s}{(1/2)_s\,S!}H^{m,n+2:IR':m_1,n_1;\dots;m_r,n_r}_{p+2,q+1:IR:p_1,q_1;\dots;p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix}(1/2+s-P-\rho;h_1,\dots,h_r),(1/2-s-\rho;k_1,\dots,k_r): \\ (-P-\rho-\rho_1;h_1+k_1,\dots,h_r+k_r):\end{matrix}\right.$$

$$\left.\begin{matrix} \left(a_j;\alpha'_j,\dots,\alpha_j^{(r)}\right)_{1,p} : \left(e_j : u'_j g'_j,\dots,u_j^{(r)} g_j^{(r)}\right)_{1,IR} : \left(c'_j,\gamma'_j\right)_{1,p_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta'_j,\dots,\beta_j^{(r)}\right)_{1,q} : \left(\ell_j : U'_j f'_j,\dots,U_j^{(r)} f_j^{(r)}\right)_{1,IR} : \left(d'_j,\delta'_j\right)_{1,q_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right] \quad \dots(5.3)$$

with the conditions as in (2.1). Putting the values of C_0 and C_p in (5.2). We obtain (4.1). Fourier sine series may also be obtained similarly.

6. Special Cases.

Case I. If we put $m=IR'=IR=0$ in (2.1) we get result in terms of multi-variable H-function defined by Srivastava and Panda [8] as :

$$\int_0^{\pi}(cos\theta/2)^{2\rho}(sin\theta/2)^{2\rho_1}H\left[(cos\theta/2)^{2h_1}(sin\theta/2)^{2k_1}z_1,\dots,(cos\theta/2)^{2h_r}(sin\theta/2)^{2k_r}z_r\right]d\theta$$

$$=H^{0,n+2:m_1,n_1;\dots;m_r,n_r}_{p+2,q+1:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \end{matrix}\middle| \begin{matrix}(1/2-\rho;h_1,\dots,h_r),(1/2-\rho_1;k_1,\dots,k_r)\left(a_j;\alpha'_j,\dots,\alpha_j^{(r)}\right)_{1,p}: \\ (-\rho-\rho_1;h_1+k_1,\dots,h_r+k_r);\qquad \left(b_j;\beta'_j,\dots,\beta_j^{(r)}\right)_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix} (c'_j;\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

provided that

1. $Re\left(2\rho+2\sum_{i=1}^{r}h_i\lambda_i+1\right)>0$,

2. $Re\left(2\rho_1+2\sum_{i=1}^{r}k_i\lambda_i+1\right)>0$ where $\lambda_i=\min_{1\le j\le m_i}\left[Re\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]$ $(i=1,2,\ldots,r)$.

Case II. Similarly if we put $m=IR'=IR=0$ in (4.1) we get the result in terms of multivariable H-function defined by Srivastava and Panda [8] as:

$$(cos\theta/2)^{2\rho}(sin\theta/2)^{2\rho_1}H\left[(cos\theta/2)^{2h_1}(sin\theta/2)^{2k_1}z_1,\ldots,(cos\theta/2)^{2h_r}(sin\theta/2)^{2k_r}z_r\right]$$

$$=\frac{1}{\pi}H^{0,n+1:m_1,n_1;\ldots;m_r,n_r}_{p+1,q+1:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(1/2-\rho;h_1,\ldots,h_r), & (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,p}: \\ (-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r), & (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]+\frac{2}{\pi}\sum_{P=1}^{\infty}\left\{\sum_{s=0}^{p}\frac{(-P)_s(-P+1/2)_s}{(1/2)_s\,s!}\right\}$$

$$H^{0,n+2:m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:p_1q_1;\ldots;p_rq_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(1/2+s-P-\rho;h_1,\ldots,h_r),(1/2-s-\rho_1;k_1,\ldots,k_r): \\ (-P-\rho-\rho_1;h_1+k_1,\ldots,h_r+k_r):\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,p}:(c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,q}:(d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right], \qquad \ldots(6.2)$$

with the conditions as in (6.1) and $P=0,1,2,\ldots$

Case III. Again in (1.4) if we put $\rho_1=0$, then the hyper-geometric function ${}_3F_2$ reduces to ${}_2F_1$. The resultant is written as:

$$ {}_2F_1(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, Re(c-a-b)>0 \text{ .} $$

On further simplification, we get

$$ \int_0^{\pi} cos\, P\theta(cos\theta/2)^{2\rho}\, d\theta=\frac{\Gamma(2\rho+1)}{2^{2\rho}\Gamma(\rho\pm P+1)}, \quad ...(6.3) $$

provided that $Re(\rho)>-1/2$ and $P=0,1,2,...$ which is known result given by MacRobert [3].

Case IV. Also if we put $\rho=0$ in (1.4) and proceed similarly as above, we obtain

$$ \int_0^{\pi} cos\, P\theta(sin\theta/2)^{2\rho_1}\, d\theta=\frac{\Gamma(2\rho_1+1)\Gamma(1/2\pm P)}{2^{2\rho_1}\Gamma(\rho_1\pm P+1)}, \quad ...(6.4) $$

provided that $Re(\rho_1)>-1/2$ and $P=0,1,2,...$ which is the known result due to Sneddon [7].

Case V. In (2.2) using $r=2$ and putting $p_1=k_1=k_2=.....=k_r=0$. we obtain a result in terms of H-function of two variables (defined by Mittal and Gupta [4]), given by koul [1,2].

Case VI. Again using $\rho_1=h_1=h_2=.....=h_1=k_1=k_2=...=k_r=0$ and taking $r=2$ in (2.2) we get the result given by Koul [1.2].

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A REMARK ON "FRACTIONAL CALCULUS OPERATOR INVOLVING THE PRODUCT OF HYPERGEOMETRIC FUNCTION OF SEVERAL VARIABLES" BY V.B.L. CHAURASIA AND HARI SINGH PARIHAR Jñānābha, 35, (2005), 175-187

By

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ABSTRACT

All the 65 results of this paper from (2.1) to (2.65) are false in view of the fact that in right hand side of each results k's run over summation signs of multiple hypergeometric series and therefore no such k should be involved in the left hand side (i.e. left hand side of each result should be free from k's.).

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PHASE SHIFTS OF S-WAVE SCHRÖDINGER EQUATION FOR MITTAG- LEFFLER FUNCTION

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ABSTRACT

In this paper, we consider a potential function to study the phase shift difference of s-wave Schrödinger equation.

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Keywords: s-wave Schrödinger equation, Phase shifts, Mittag-Leffler function.

1. Introduction. The phase shifts have great importance in the computation of scattering and experimental work of nuclear and atomic collision. Several mathematical problems are solved to find out a potential function from a observed phase shift difference. Many workers namely Mahajan and Varma [8], Raghuwansi and Sharma [9], Kumar, Chandel and Agrawal [5], Agrawal and Kumar [1], Chandel and Kumar [3] have determined the phase shifts from a given potential function. Recently, Kumar and Singh [6] have obtained an approximation formula for phase-shifts of s-wave Schrödinger equation with the application of binomial potential function and study the phase-shifts variation with respect to the parameter involving in binomial potential function.

Here, in our work, we consider a potential function of Mittag- Leffler function and then on applying Tietz method [12], we obtain phase shift difference in series form involving hypergeometric function [8] and then make some studies on shift variation verses the parameters involving in the potential function.

The Mittag- Leffler function (see Erdélyi et al. [4] and Srivastava and Manocha [11]) is defined by

$$E_a(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(an+1)}, a\in C \qquad ...(1.1)$$

We consider the potential function in the form

$$U(r)=E_a\left(e^{br}\right), \qquad a,b\in C. \qquad \ldots(1.2)$$

Particularly, on setting $a=0$ in (1.2), we find the potential function of Bhattacharjie and Sudarshan [2] in the form

$$U'(r)=U(r)-1=\sum_{n=1}^{\infty}e^{nbr}. \qquad \ldots(1.3)$$

Also, from (1.3), we get

$$\frac{dU'(r)}{dr}=\frac{dU(r)}{dr}=\sum_{n=1}^{\infty}A_n e^{nbr} \text{ at } a=0. \qquad \ldots(1.4)$$

The potential function considered in (1.2) have more parameters to that of Bhattacharjie-Sudarshan [2] function and Chandel-Kumar [3] function. In this paper, we determine the phase shift difference formula for this function on making use of the Tietz' s techniques and make some studies on shift variation with respect to the parameters involving in the parameters given in the potential function (1.2).

2. Formulae Used. In our investigation, we need the applications of following formulae :

For the s-wave radial Schrödinger equation

$$\frac{d^2\Psi(r)}{dr^2}+\left[K^2-U(r)-\frac{L(L+1)}{r^2}\right]\Psi_{(r)}=0. \qquad \ldots(2.1)$$

Using Tietz [12] method and the Luke [7] formulae, Chandel and Kumar [3] have given the phase shifts in the form

$$\eta_L-\eta_{L+1}=\frac{\pi K^{2L+1}}{2^{2L+3}\Gamma(L+3/2)\Gamma(L+5/2)}\int_0^{\infty}\frac{dU}{dr}r^{2L+1}\,{}_1F_2\left[\begin{matrix}L+2; \\ L+5/2,2L+3;\end{matrix}\; -K^2r^2\right]dr,$$

provided that $(2L+3)>0$ ...(2.2)

The second phase difference formula is given by

$$\eta_{L-1}-\eta_{L+1}=\frac{\pi K^{2L+1}}{2^{2L+3}\Gamma(L+1/2)\Gamma(L+3/2)}\int_0^{\infty}\frac{dU}{dr}r^{2L+3}\,{}_1F_2\left[\begin{matrix}L+1; \\ L+3/2,2L+2;\end{matrix}\; -K^2r^2\right]dr \quad (2.3)$$

3. Phase Shift Difference for Mittag-Leffler Function . From (2.1), we find

$$\frac{dU}{dr}=\frac{b}{a}\sum_{n=1}^{\infty}\frac{e^{nbr}}{\Gamma(an)} \qquad \ldots(3.1)$$

Now, making an appeal to (3.1) and (2.2), we derive

$$\eta_L - \eta_{L+1} = \frac{\sqrt{\pi}K^{2L+1}\Gamma(L+2)}{ab^{2L+3}\Gamma(L+3/2)}\sum_{n=1}^{\infty}\frac{1}{n^{2L+4}\Gamma(an)}\,{}_2F_1\left[\begin{matrix}L+2, L+2; & \dfrac{-4K^2}{n^2b^2}\\ 2L+3; & \end{matrix}\right],$$

provided that $\left|(2iK/nb)^2\right| < 1$ . ...(3.2)

Further, Making an appeal to (3.1) and (3.2), we get

$$\eta_{L-1} - \eta_{L+1} = \frac{\sqrt{\pi}K^{2L-1}\Gamma(L+1)}{ab^{2L+1}\Gamma(L+1/2)}\sum_{n=1}^{\infty}\frac{1}{n^{2L+2}\Gamma(an)}\,{}_2F_1\left[\begin{matrix}L+1, L+1; & \dfrac{-4K^2}{n^2b^2}\\ 2L+2; & \end{matrix}\right],$$

provided that $\left|(2iK/nb)^2\right| < 1$. ...(3.3)

4. Particular Cases . For $L=0$, (3.2), gives

$$\eta_0 - \eta_1 = \frac{\sqrt{\pi}K}{ab^3\Gamma(3/2)}\sum_{n=1}^{\infty}\frac{1}{n^4\Gamma(an)}\,{}_2F_1\left[\begin{matrix}2,2; & \dfrac{-4K^2}{n^2b^2}\\ 3; & \end{matrix}\right]. \qquad ...(4.1)$$

while for $L=1$, (3.2) gives

$$\eta_1 - \eta_2 = \frac{2\sqrt{\pi}K^3}{ab^5\Gamma(5/2)}\sum_{n=1}^{\infty}\frac{1}{n^6\Gamma(an)}\,{}_2F_1\left[\begin{matrix}3,3; & \dfrac{-4K^2}{n^2b^2}\\ 5; & \end{matrix}\right]. \qquad ...(4.2)$$

From (4.2), we may write

$$\eta_1 - \eta_2 = \frac{\sqrt{\pi}K}{ab^3\Gamma(3/2)}\sum_{n=1}^{\infty}\frac{1}{n^4\Gamma(an)}\sum_{s=1}^{\infty}\frac{s(2)_s(2)_s}{3(4)_s}\frac{\left(-4K^2/n^2b^2\right)^s}{s!}. \qquad ...(4.3)$$

Now, adding (4.1) and (4.3) we can write

$$\eta_0 - \eta_2 = \frac{\sqrt{\pi}K}{ab^3\Gamma(3/2)}\sum_{n=1}^{\infty}\frac{1}{(n)^4\,\Gamma(an)}\,{}_2F_1\left[\begin{matrix}2,2; & \dfrac{-4K^2}{n^2b^2}\\ 4; & \end{matrix}\right]. \qquad ...(4.4)$$

If we put $L=1$ in the equation (3.3), then we get directly above result (4.4).

5. Analysis. To analize above results we start with

Table No. 1

a	$\eta_0 - \eta_2$
1	22.9226033
2	11.40945107
3	3.800312649
4	0.950005558

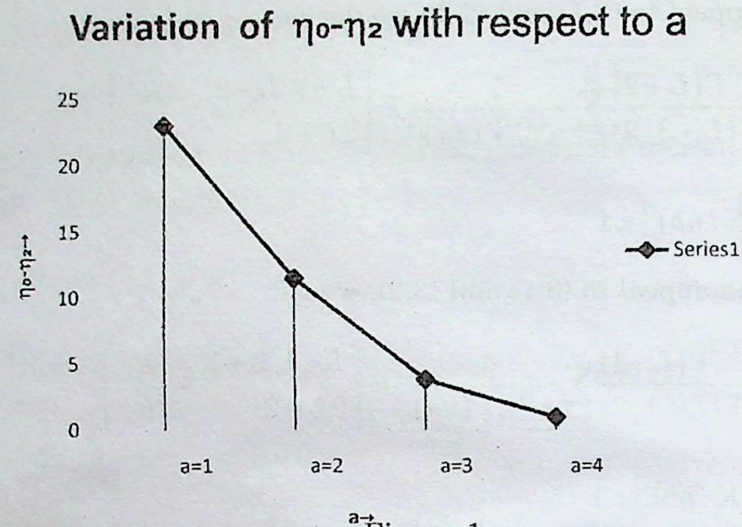

Figure 1.

The above graph shows that when potential function Figune vo.1 woar is reduced on making increment of a so that internal kinetic energy of the particle is increased and by this kinetic energy the particle of s-orbit gains frequency of d-orbit and then phase of s-wave tends to phase of d-wave ($\eta_0 - \eta_2$), thus difference in phase in sharply reducing.

Table No. 2

b	$\eta_0 - \eta_2$
1	22.9226033
2	0.239139705
3	0.085412647
4	0.027040942

Variation η_0–η_2 of with respect to *b*.

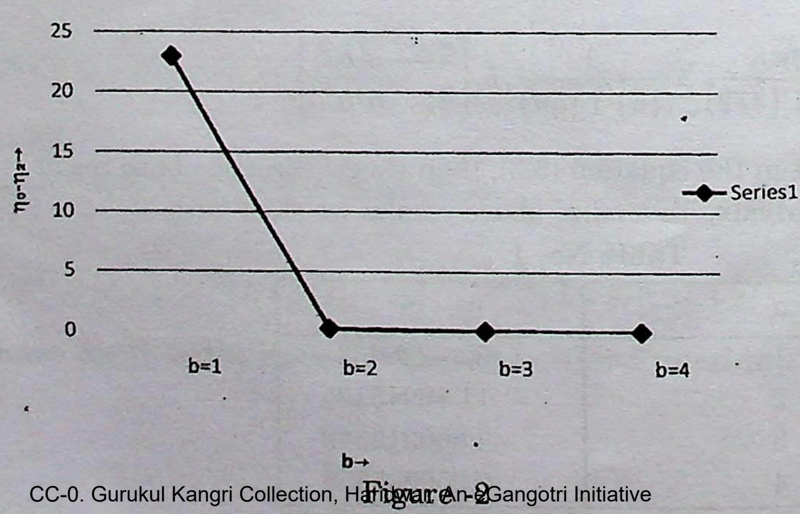

Figure 2

By second graph it is shown that when we make the increment in b, the potential energy of the particle is grown up, as when as the potential energy is changed, the internal energy of the particle is also changed so that the particle is disturbed due to slightly scattering and thus phase shifts is sharply decreased but when potential energy becomes large the difference in phase shifts becomes constant and the particle remains is *s*-orbit.

Table No. 3

b	$\eta_0 - \eta_2$
1/2	3458.01349
1/3	61372.55374
1/4	465550.4809
1/5	2233824.794

Variation of $\eta_0 - \eta_2$ with respect to *b*.

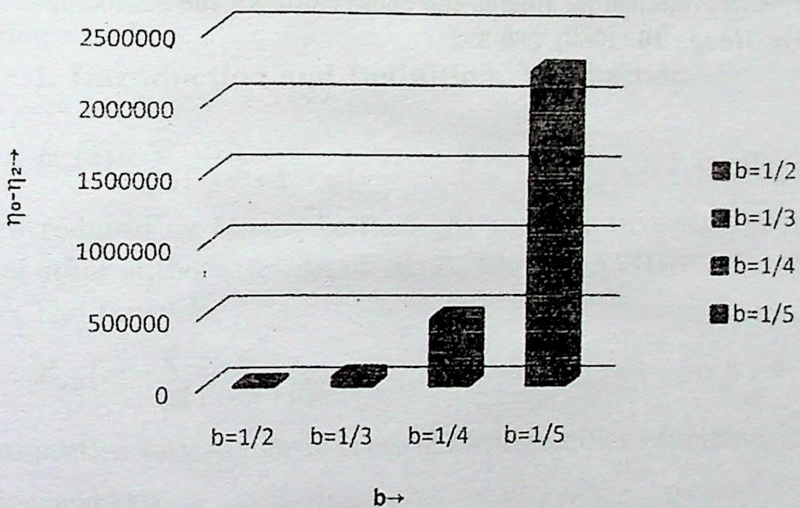

Figure. 3

From third graph, we have that when we reduce the value of b in the formula (1.2), the potential energy of the particle is reduced and thus internal kinetic energy of the particle is increased and by this effect the particle is scattered so that the difference in phase is sharply increasing.

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A SHORT NOTE ON EXTON'S RESULT

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ABSTRACT

In this paper, authors derived some generating functions (partly bilateral and partly unilateral) involving exponential and Mittag-Leffer's functions in view of Exton's result [3].

2000 Mathematics Subject Classification : 33E99

Keywords: Mittag-Leffler's function and related function, Hypergeometric function.

1. Introduction and Definition. The function

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}, \qquad \alpha>0 \tag{1.1}$$

was introduced by Mittag-Leffler's [5] and was investigated systimatically by several other authors (for detail, see [2, Chapter XVIII]).

The function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \qquad \alpha,\beta>0 \tag{1.2}$$

has properties very similar to those of Mittag-Leffler's function $E_{\alpha}(z)$ (See Wiman [9], Agarwal [1]).

In 1971, Prabhakar [7] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_{k}}{\Gamma(\alpha k+\beta)}\frac{z^{k}}{k!}, \qquad \alpha,\beta,\gamma>0 \tag{1.3}$$

where $(\gamma)_k$ is the Pochhammer symbol (Rainville [8])

$$(\gamma)_0=1,(\gamma)_k=\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+k-1).$$

The function $E_{\alpha,\beta}^{\gamma}(z)$ is most natural generalization of the exponential function $exp(z)$, Mittag-Leffler function $E_{\alpha}(z)$ and Wiman's function $E_{\alpha,\beta}(z)$.

We note that

$$
\left.\begin{array}{l}
E_{\alpha,\beta}^{1}(z)=E_{\alpha,\beta}(z), E_{\alpha,1}(z)=E_{\alpha}(z), E_{1,\beta}(z)=\dfrac{1}{\Gamma\beta}\,{}_1F_1[1,\beta;z] \\
E_{1,1}^{1}(z)=E_{1,1}(z)=E_1(z)=e^{z}, E_2(z^2)=\cosh z
\end{array}\right\} \qquad (1.4)
$$

An interesting (partly bilateral and partly unilateral) generating function for $F_n^m(x)$, due to Exton [3, p.147(3)] is recalled here in the following (modified) from [see [6]):

$$
exp\left(s+t-\frac{xt}{s}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n F_n^m(x), \qquad (1.5)
$$

where $F_n^m(x)={}_1F_1(-n;m+1;x)/m!n!=L_n^m(x)/(m+n)!,$ (1.6)

and $L_n^m(x)$ denotes the classical Laguerre polynomials, (see [8] and in what folllows

$$
m^*=max(0,-m),(m\varepsilon Z=0,1,2,...) \qquad (1.7)
$$

so that all factorials in equation (1.5) have meaning.

2. Generating Reltions.

Result-1. If $p,q,l\in N$, then

$$
exp\left(s^p+t^q-\left(\frac{xt}{s}\right)^l\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n \sum_{r=0}^{[n/q]}\frac{(-x)^{lr}}{r!\left(\dfrac{m+lr}{p}\right)!\left(\dfrac{n-lr}{q}\right)!} \qquad (2.1)
$$

Special Cases

(i) For $p=q=l$, equation (2.1) reduces to

$$
exp\left(s^p+t^p-\left(\frac{xt}{s}\right)^p\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{(m/p)!(n/p)!}\,{}_1F_1\left[\begin{array}{ll}-n/p & ;\\ m/p+1 & ;\end{array} -(-x)^p\right]. \qquad (2.2)
$$

(ii) When $p=2$ in (2.2) or $p=q=l=2$ in equation (2.1), we have

$$
exp\left(s^2+t^2-\left(\frac{xt}{s}\right)^2\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{(m/2)!(n/2)!}\,{}_1F_1\left[\begin{array}{ll}-n/2 & ;\\ m/2+1 & ;\end{array} x^2\right]. \qquad (2.3)
$$

(iii) When $p=q=1$ in equation (2.1), we get

$$
exp\left(s+t-(xt/s)^l\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{m!n!}\,{}_1F_1\left[\begin{array}{ll}\prod_{j=1}^{l}\dfrac{-n+j-1}{l} & ;\\ & \quad x^l \\ \prod_{j=1}^{l}\dfrac{(m+1)+j-1}{l}+1 & ;\end{array}\right]. \qquad (2.4)
$$

For $l=1$, equation (2.4) reduces to (1.5). It can also be obtained from (2.1) by

taking $p=q=l=1$.

Result-2. If $p,q,l \in N$ and $E^{\gamma}_{\alpha,\beta}$ is defined by (1.3), then

$$E^{\gamma_1}_{\alpha_1,\beta_1}\left(s^p\right)E^{\gamma_2}_{\alpha_2,\beta_2}\left(t^q\right)E^{\gamma_3}_{\alpha_3,\beta_3}\left(-xt/s\right)^l$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/q]}\frac{(\gamma_1)_{\left(\frac{m+lr}{p}\right)}(\gamma_2)_{\left(\frac{n-lr}{q}\right)}(\gamma_3)_r(-x)^{lr}}{(\beta_1)_{\alpha_1\left(\frac{m+lr}{p}\right)}(\beta_2)_{\alpha_2\left(\frac{n-lr}{q}\right)}(\beta_3)_{\alpha_3 r}}. \qquad (2.5)$$

Special Cases.

(i) For $\gamma_1 = \gamma_2 = \gamma_3 = 1$, equation (2.5) reduces to

$$E_{\alpha_1,\beta_1}\left(s^p\right)E_{\alpha_2,\beta_2}\left(t^q\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^l$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/q]}\frac{(-x)^{lr}}{(\beta_1)_{\alpha_1\left(\frac{m+lr}{p}\right)}(\beta_2)_{\alpha_2\left(\frac{n-lr}{q}\right)}(\beta_3)_{\alpha_3 r}}. \qquad (2.6)$$

(ii) When $p=q=l$ in equation (2.5), we get

$$E_{\alpha_1,\beta_1}\left(s^p\right)E_{\alpha_2,\beta_2}\left(t^q\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^p$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/p]}\frac{(-x)^{pr}}{(\beta_1)_{\alpha_1\left(\frac{m}{p}+r\right)}(\beta_2)_{\alpha_2\left(\frac{n}{p}-r\right)}(\beta_3)_{\alpha_3 r}}. \qquad (2.7)$$

For $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 1$ equation (2.7) reduces to (2.2) and also to (1.5) for $p=1$.

(iii) If $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and $p = q = l = 1$ in equation (2.5), we get known generating function of Kamarujjama and Khursheed [4]

$$E_{\alpha_1,\beta_1}(s)E_{\alpha_2,\beta_2}(t)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{n}\frac{(-x)^{r}}{(\beta_1)_{\alpha_1(m+r)}(\beta_2)_{\alpha_2(n-r)}(\beta_3)_{\alpha_3 r}}. \qquad (2.8)$$

(iv) When $p=2$ in equation (2.7) or $p=q=l=2$ in equation (2.6), we get

$$E_{\alpha_1,\beta_1}\left(s^2\right)E_{\alpha_2,\beta_2}\left(t^2\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^2$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/2]}\frac{(-x)^{2r}}{(\beta_1)_{\alpha_1\left(\frac{m}{2}+r\right)}(\beta_2)_{\alpha_2\left(\frac{n}{2}-r\right)}(\beta_3)_{\alpha_3 r}}. \qquad (2.9)$$

For $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 1$ equation (2.9) reduces to (2.3).

Now putting $\alpha_1 = \alpha_2 = \alpha_3 = 2, \beta_1 = \beta_2 = \beta_3 = 1$ in equation (2.9) and using a relation (1.4), we obtain a generating function of hypergeometric function ${}_2F_3$ in terms of hyperbolic cosine functions.

$$\cosh s \cosh t \cosh(xt/s) = \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty} \frac{s^{2m}t^{2n}}{(2m)!(2n)!}\, {}_2F_3\left[\begin{matrix}-n,-n+1/2 & ; \\ m+1, m+1/2, 1/2 & ;\end{matrix} x^2/4\right]. \quad (2.10)$$

(v) For $p = q = 1$, equation (2.6) reduces to

$$E_{\alpha_1,\beta_1}(s)E_{\alpha_2,\beta_2}(t)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^l$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty} \frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3} \sum_{r=0}^{[n/l]} \frac{(-x)^{lr}}{(\beta_1)_{\alpha_1(m+lr)}(\beta_2)_{\alpha_2(n-lr)}(\beta_3)_{\alpha_3 r}}. \quad (2.11)$$

For $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 1$, equation (2.11) reuces to (2.4) and also to (1.5) for $l=1$.

(vi) For $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $l=1$, equation (2.11) reduces to a following generating relation

$${}_1F_1[1,\beta_1;s]\,.\,{}_1F_1[1,\beta_2;t]\,.\,{}_1F_1[1,\beta_3;-xt/s] = \sum_{m=-\infty}^{\infty}\sum_{n=m^{\circ}}^{\infty} \frac{s^m t^n}{(\beta_1)_m(\beta_2)_n}\, {}_2F_2\left[\begin{matrix}1, 1-\beta_2-n & ; \\ \beta_1+m, \beta_3 & ;\end{matrix} x\right] \quad (2.12)$$

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APPLICATIONS OF SAIGO FRACTIONAL CALCULUS OPERATORS FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

In the present paper, we introduce the subclass $S_j(n,p,\lambda,q,\alpha)$ of functions with negative coefficients, which are analytic and multivalent in the unit disc $U=\{z:|z|<1\}$. The fractional calculus of functions associated with ingegral operator $J_{c,p}$ in the class $S_j(n,p,\lambda,q,\alpha)$ as applications of the Saigo fractional calculus operator $I_{0,z}^{\beta,\gamma,\eta}$ are established here.

Corresponding to our main theorems some known and unkown results for the multivalent functions are also shown to be deduced as the special cases.

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1. Introduction. Let $T(j,p)$ be the class of function of the form

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_k z^k \qquad \left(a_k \geq 0; p,j \in N=\{1,2,\ldots\}\right), \tag{1}$$

which are analytic and p-valent in the open disc $U=\{z:|z|<1\}$. A function $f(z)\in T(j,p)$ is said to be p-valently close to convex of order α in U if it satisfies the inequality

$$Re\left\{z^{1-p} f(z)\right\} > \alpha \qquad (z \in U; 0 \le \alpha < p; p \in N). \tag{2}$$

A function $f(z) \in T(j,p)$ is said to be p-valently starlike of order α in U if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U; 0 \le \alpha < p; p \in N). \tag{3}$$

Further more a function $f(z) \in T(j,p)$ is said to p-valently convex of order α in U if it satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U; 0 \le \alpha < p; p \in N). \tag{4}$$

For each $f(z) \in T(j,p)$ we have [2]

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q} \quad \left(q \in N_0 = N \cup \{0\}; p > q\right). \tag{5}$$

For a function $f(z)$ in class $T(j,p)$, we define the following differential operator:

$$D^0_{p,\lambda} f^{(q)}(z) = f^{(q)}(z),$$

$$D^1_{p,\lambda} f^{(q)}(z) = Df^{(q)}(z) = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda} f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{(k+\lambda-q)}{(p+\lambda-q)} \frac{k!}{(k-q)!} a_k z^{k-q},$$

$$D^2_{p,\lambda} f^{(q)}(z) = D\left[D^1_{p,\lambda} f^{(q)}(z)\right] = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda} D^1_{p,\lambda} f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q}\right)^2 a_k z^{k-q},$$

and so on

$$D^n_{p,\lambda} f^{(q)}(z) = D\left[D^{n-1}_{p,\lambda} f^{(q)}(z)\right] = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda} D^{n-1}_{p,\lambda} f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^n a_k z^{k-q}$$

$$(p, j \in N; q \in N_0; p > q; \lambda \geq 0) \tag{6}$$

A function $f^{(q)}(z) \in T(j,p)$ is said to be in $S_j(n,p,\lambda,q,\alpha)$ if and only if

$$Re\left(\frac{z\left[D^n_{p,\lambda} f^{(q)}(z)\right]'}{D^n_{p,\lambda} f^{(q)}(z)} \right) > \alpha; \tag{7}$$

where $z \in U, \lambda \geq 0, p \in N, q, n \in N_0, 0 \leq \alpha < p-q, p > q$ and $D^n_{p,\lambda}$ is defined in (6).

We note that for $q = 0$ the operator $D^n_{p,\lambda} f(z)$ in view of multiplier transformation is defined and studied recently by Agharaly et al. [1] and Singh et al. [13] for positive coefficients in the following form

$$I_p(n,\lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^n a_k z^k \qquad (\lambda \geq 0, n \in Z).$$

Earlier the operator $I_1(n,\lambda)$ was investigated by Cho and Srivastava [6] and Cho and Kim [7]. Whereas the operator $I_1(n,l)$ was studied by Uralegaddi and Somanatha [16]. $I_1(n,0)$ is the well known *Sălăgean* derivative operator [11] defined as

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = NU\{0\}.$$

2. Coefficient Estimate.

Theorem 1. Let the function $f(z)$ defined by (1) be in the class $T(j,p)$. Then the function $f(z)$ belongs to the class $S_j(n,p,\lambda,q,\alpha)$ if and only if

$$\sum_{k=j+p}^{\infty} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^n (k-q-\alpha)\delta(k,q) a_k \leq (p-q-\alpha)\delta(p,q) \tag{8}$$

$$(0 \leq \alpha < p-q; p, j \in N; q, n \in N_0; p > q; \lambda \geq 0;)$$

where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases}p(p-1)\ldots(p-q+1), & q\neq 0\\ 1 & ,q=0\end{cases} \tag{9}$$

Proof. Let us suppose the inequality (8) holds true. Then in view of (7), we have

$$\left|\frac{z\left[D_{p,\lambda}^{n}f^{(q)}(z)\right]'}{D_{p,\lambda}^{n}f^{(q)}(z)}-(p-q)\right|\leq\frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}(k-p)\delta(k,q)a_{k}|z|^{k-p}}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}\delta(k,q)a_{k}|z|^{k-p}}$$

$$\leq\frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}(k-p)\delta(k,q)a_{k}}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}\delta(k,q)a_{k}}\leq p-q-\alpha$$

Therefore the values of function

$$\phi(z)=\frac{z\left[D_{p,\lambda}^{n}f^{(q)}(z)\right]}{D_{p,\lambda}^{n}f^{(q)}(z)} \tag{10}$$

lie in a circle which is centered at $w=(p-q)$ and whose radius is $(p-q-\alpha)$. Hence the function $f(z)$ satisfies the condition given in (7)

Now conversely, assume that the function $f(z)$ is in the class $S_{j}(n,p,\lambda,q,\alpha)$. Then we have

$$Re\left\{\frac{z\left[D_{p,\lambda}^{n}f^{(q)}(z)\right]}{D_{p,\lambda}^{n}f^{(q)}(z)}\right\}=Re\left\{\frac{(p-q)\delta(p,q)-\sum\limits_{k=j+p}^{\infty}(k-q)\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}\delta(k,q)a_{k}z^{k-q}}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}\delta(k,q)a_{k}z^{k-q}}\right\}>\alpha \tag{11}$$

for some $\alpha(0\leq\alpha<p-q;p,j\in N;q,n\in N_{0};p>q;\lambda\geq 0)$ and $z\in U$. Choose value of z on the real axis so that $\phi(z)$ given by (10) is real. Upon clearing the denominator in (11) and letting $z\rightarrow 1^{-}$ through the real values we can see that

$$(p-q)\delta(p,q)-\sum_{k=j+p}^{\infty}(k-q)\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^{n}\delta(k,q)$$

$$a_k \geq \alpha \left\{ \delta(p,q) - \sum_{k=j+p}^{\infty} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^n \delta(k,q) a_k \right\}, \tag{12}$$

which leads to inequality (8). It completes the proof of Theorem 1.

Corollary 1. Let the function $f(z)$ defined by (1) be in the class $S_j(n,p,\lambda,q,\alpha)$ then the following inequality hold true

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k+\lambda-q}{p+\lambda-q} \right)^n (k-q-\alpha)\delta(k,q)} \tag{13}$$

$$(k \geq j+p; p,j \in N; q,n \in N_0; \lambda \geq 0; p > q).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k+\lambda-q}{p+\lambda-q} \right)^n (k-q-\alpha)\delta(k,q)} z^k \tag{14}$$

$$(k \geq j+p; p,j \in N; q,n \in N_0; \lambda \geq 0; p > q).$$

3. Applicatrions of Fractional Calculus. In our present investigation, we shall make use of the familiar integral operator $J_{c,p}$ defined by [5,p.676, eq.(1.8)]

$$\left(J_{c,p}f\right)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\,dt \qquad \left(f \in T(j,p); c > -p; p \in N\right). \tag{15}$$

Definition 1. The Reimann-Liouville fractional integral of order λ is defined, for a function $f(z)$, by [15,p.224,eq.(3.1)]

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{16}$$

where $\lambda > 0, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and hte multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log\,(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. The Reimann-Liouville fractional derivative of order λ is defined,

for a function $f(z)$, by [15,p.224, eq. (3.2)]

$$D_z^{\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta, \tag{17}$$

where $0 \le \lambda < 1, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed as in Definition 1 above.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by [15,p.225, eq. (3.3)]

$$D_z^{(n+\lambda)} f(z)=\frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{18}$$

where $0 \le \lambda < 1$ and $n \in N_0 = NU\{0\}$.

Srivastava, Saigo and Owa defined the following fractional integral operator involving Gauss's hypergeometric function:

Definition-4. For real number $\alpha > 0, \beta$ and η, the Saigo fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by [12,p.112, Eq. (8)] (See also [15]):

$$I_{0,z}^{\alpha,\beta,\eta} f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta}{z}\right) f(\zeta)d\zeta, \tag{19}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$f(z)=O\left(|z|^{\epsilon}\right)$, as $z \to 0$,

where $\epsilon > max\{0,\beta-\eta\}-1$ and the many-valuedness of $(z-\xi)^{\alpha-1}$ is removed by requiring $log(z-\xi)$ to be real when $(z-\xi)>0$.

From Definition (1) and Definition (4), it is easy to see that

$D_z^{-\alpha} f(z)=I_{0,z}^{\alpha,\alpha,\eta} f(z)$.

Lemma. If $\alpha > 0$ and $k > \beta-\eta-1$, then [12]

$$I_{0,z}^{\alpha,\beta,\eta} z^k=\frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta} \tag{20}$$

For a function $f(z) \in T(j,p)$ defined by (1), we obtain easily the following results in view of (15) and (20) respectively :

$$\left(J_{c,p}f\right)(z) = z^p - \sum_{k=j+p}^{\infty} \left(\frac{c+p}{c+k}\right) a_k z^k \qquad (c > -p; p, j \in N) \tag{21}$$

and

$$I_{0,z}^{\beta,\gamma,\eta}\{f(z)\} = \frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} z^{p-\gamma} - \sum_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k z^{k-\gamma} \tag{22}$$

Now for the function $\left(J_{c,p}f\right)(z)$ defined in (21) we obtain the following fractional integral in view of (22)

$$I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\} = \frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} z^{p-\gamma} -$$

$$\sum_{k=j+p}^{\infty} \left(\frac{c+p}{c+k}\right) \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k z^{k-\gamma} \tag{23}$$

$$(\beta > 0, c > -p; p, j \in N) .$$

Theorem 2. Let β, γ and η satisfy inequalities $\beta > 0, \gamma < p+1, \gamma - \eta < p+1$, $\beta + \eta > -(p+1)$. Choose a positive integer such that $n \geq \gamma(\beta+\eta)/\beta - p - 1$.

If $f(z) \in S_j(n,p,\lambda,q,\alpha)$ then

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} - \frac{(c+p)}{c+p+j}\right.$$

$$\left.\frac{\Gamma(p+j+1)\Gamma(p+j-\gamma+\eta+1)(p-q-\alpha)\delta(p,q)}{\Gamma(p+j-\gamma+1)\Gamma(p+j+\beta+\eta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)\delta(j+p,q)} |z|^j \right\} |z|^{p-\gamma}$$

$$(z \in U_0; 0 \leq \alpha < p-q; c > -p; p, j \in N; q, n \in N_0; p > q; \lambda \geq 0) \tag{24}$$

and

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} + \frac{(c+p)}{c+p+j}\right.$$

$$\left.\frac{\Gamma(p+j+1)\Gamma(p+j-\gamma+\eta+1)(p-q-\alpha)\delta(p,q)}{\Gamma(p+j-\gamma+1)\Gamma(p+j+\beta+\eta+1)\left(\dfrac{j+p+\lambda-q}{p+\lambda-q}\right)^{n}(j+p-q-\alpha)\delta(j+p,q)}|z|^{j}\right\}|z|^{p-\gamma}$$

$$\left(z\in U_0;0\le\alpha<p-q;c>-p;p,j\in N;q,n\in N_0;p>q;\lambda\ge 0\right), \tag{25}$$

where

$$U_0=\begin{cases}U, & \gamma\le p\\ U-\{0\}, & \gamma>p\end{cases}.$$

These results are sharp for function $f(z)$ given by

$$\left(J_{c,p}f\right)(z)=z^{p}-\frac{(c+p)(p-q-\alpha)\delta(p,q)}{(c+p+j)\left(\dfrac{j+p+\lambda-q}{p+\lambda-q}\right)^{n}(j+p-q-\alpha)\delta(j+p,q)}z^{j+p}. \tag{26}$$

Proof. To prove the Theorem 2, we start from eq. (23), i.e.

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right|=\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}z^{p-\gamma}-$$

$$\sum_{k=j+p}^{\infty}\left(\frac{c+p}{c+k}\right)\frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}a_k z^{k-\gamma}. \tag{27}$$

Now on setting

$$H(z)=\frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}z^{\gamma}\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \tag{28}$$

the above result given in (27) takes the following form

$$H(z)=z^{p}-\sum_{k=j+p}^{\infty}a_k\psi(k)z^{k} \tag{29}$$

where

$$\psi(k)=\frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}\frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}\frac{(c+p)}{(c+k)}.$$

Since $\psi(k)$ is a decreasing function of k, therefore

$$\psi(k) \le \psi(p+j)$$

$$= \frac{\Gamma(j+p+1)\Gamma(j+p-\gamma+\eta+1)}{\Gamma(j+p-\gamma+1)\Gamma(j+p+\beta+\eta+1)} \frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)} \frac{(c+p)}{(c+p+j)} \quad (30)$$

Now in view of Theorem 1, we have

$$\sum_{k=j+p}^{\infty} a_k \le \frac{(p-q-\alpha)\delta(p,q)}{\delta(j+p,q)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)}. \quad (31)$$

Hence for the function $H(z)$ obtained in (29), we have

$$|H(z)| \ge |z|^p - |z|^{j+p}\,\psi(j+p)\sum_{k=j+p}^{\infty} a_k .$$

Therefore, making an appeal to (30) and (31), we obtain

$$|H(z)| \ge |z|^p - \frac{(c+p)}{(c+p+j)} \frac{\Gamma(j+p+1)\Gamma(j+p-\gamma+\eta+1)}{\Gamma(j+p-\gamma+1)\Gamma(j+p+\beta+\eta+1)} \frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}$$

$$\frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)\delta(j+p,q)} |z|^{j+p}.$$

Now in view of (28), we atonce arrive at the desired result in (24).
The second result of Theorem 2 is proved similarly in view of (30) and (31) for inequality

$$|H(z)| \le |z|^p + |z|^{j+p}\,\psi(j+p)\sum_{k=j+p}^{\infty} a_k .$$

4. Special Cases. (I) If in differential operator $D^n_{p,\lambda} f^{(q)}(z)$ defined in (6) we consider $\lambda = 0$ then it reduces to the operator studied by Aouf [2]. Therefore, at $\lambda = 0$ all the results established in the Sections-2, 3 will provide the known results of Aouf [2].

(II) The results obtained in class $S_j(n,p,\lambda,q,\alpha)$ here in turn provide many known results studied in various subclasses by specializing the parameters j,n,p,q,α,λ. To illustrate we give few classes as follows:

(i) $S_j(n,p,0,q,\alpha)=S_j(n,p,q,\alpha)$ (Aouf [2]),

(ii) $S_j(0,p,0,q,\alpha)=S_j(p,q,\alpha)$ and $S_j(1,p,0,q,\alpha)=C_j(p,q,\alpha)$ (Chen et al.[4]),

(iii) $S_j(n,1,0,0,\alpha)=P(j,\alpha,n)(j\in N;n\in N_0;0\le\alpha<1)$ (Aouf and Srivastava [3]),

(iv) $S_1(n,1,0,0,\alpha)=T(n,\alpha)(n\in N_0;0\le\alpha<1)$ (Hur and Oh [8]),

(v) $S_j(0,p,0,0,\alpha)=\begin{cases}T_j^*(p,\alpha) & (Owa[10])\\ T_\alpha(p,j) & (Yamakawa[17])\end{cases}$,

(vi) $S_j(1,p,0,0,\alpha)=\begin{cases}C_j^*(p,\alpha) & (Owa[10])\\ CT_\alpha(p,j) & (Yamakawa[17])\end{cases}$,

(vii) $S_1(0,p,0,0,\alpha)=T^*(p,\alpha)$ and $S_1(1,p,0,0,\alpha)=C(p,\alpha)(p\in N;0\le\alpha<p)$ (Owa [9] and Salagean et al. [11]),

(viii) $S_j(0,1,0,0,\alpha)=T_\alpha(j)$ and $S_j(1,1,0,0,\alpha)=C_\alpha(j)(n\in N_0;0\le\alpha<1)$ (Srivastava et al. [14]),

(ix) $S_j(n,p,0,0,\alpha)=S_j(n,p,\alpha)(p,j\in N,n\in N_0;0\le\alpha<p)$ (Aouf [2]),

(III) If in (24) and (25) we take $\gamma=-\beta$ then these inequalities reduce to the following inequalities involving *R-L* fractional integral operator defined in (16).

Corollary 2. Let the function $f(z)$ defined by (1) be in the class $S_j(n,p,\lambda,q,\alpha)$. Then

$$D_z^{-\beta}\{(J_{c,p}f)(z)\}\ge\left\{\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}-\frac{(c+p)\Gamma(p+j+1)(p-q-\alpha)(\delta(p,q))}{(c+p+j)\Gamma(p+j+\beta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p+\beta} \quad (32)$$

$(z\in U;0\le\alpha<p-q;\beta>0;c>-p;p,j\in N;n\in N_0;p>q;\lambda>0)$,

and

$$D_z^{-\beta}\left\{\left(J_{c,p}f\right)(z)\right\} \le \left\{\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}+\right.$$

$$\left.\frac{(c+p)\Gamma(p+j+1)(p-q-\alpha)(\delta(p,q))}{(c+p+j)\Gamma(p+j+\beta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^{n}(j+p-q-\alpha)\delta(j+p,q)}|z|^{j}\right\}|z|^{p+\beta} \quad (33)$$

$\left(z\in U;0\le\alpha<p-q;\beta>0;c>-p;p,j\in N;n\in N_0;p>q;\lambda>0\right)$.

Each of the assertion (32) and (33) is sharp for the function $f(z)$ givey by (26). The results in (32) and (33) in turn at $\lambda=0$ give the known results due to Aouf [2, p.32, eq. (6.6) and eq. (6.7)].

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SOME RESULTS ON A τ-GENERALIZED RIEMANN ZETA FUNCTION

By

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ABSTRACT

In this paper we give a τ-generalization of the new zeta function due to Goyal and Laddha. This function is defined by series and the corresponding integral representations are established. We have also investigated the properties of the new type of generating functions such as integral representations and generating functions. Several interesting results obtained earlier by Goyal and Laddha [3], Katsurada [4] and Bin-Saad [1] follow special cases of our main findings.

2010 Mathematics Subject Classification : 11M35, 11S23.

Keywords: τ–generalized zeta function, integral representation, bionomial theorem, and gamma function.

1. Introduction. An interesting definition of the zeta functions, due to Goyal and Laddha [3], is as follows:

$$\phi^{\circ}_{\mu}(y;z,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n\, y^n}{(a+n)^z\, n!},\ |y|<1, \tag{1.1}$$

$\text{Re}\,a>0, \mu\geq 1$, where $(\lambda)_n=\dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$.

This is expressed as the integral form

$$\phi^{\circ}_{\mu}(y,z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-\mu}dt\,. \tag{1.2}$$

This function is continued to a meromorphic function over the whole z-plane (cf. Erdélyi [2]). Obviously, when $\mu=1$, (1.1) reduces to the zeta function studied by Erdélyi [2]. Katsurada [4] introduced two hypergeometric type generating functions of the Riemann zeta function as follows:

$$e_z(x)=\sum_{m=0}^{\infty}\zeta(z+m)\frac{x^m}{m!},\qquad |x|<\infty, \tag{1.3}$$

and

$$f_z(v;x) = \sum_{m=0}^{\infty} (v)_m \zeta(z+m) \frac{x^m}{m!}, \qquad |x| < 1 \tag{1.4}$$

where v and z are arbitrary fixed complex parameters.
Recently, Bin-Saad [1] has defined two new type of generating functions suggested by (1.3) and (1.4) as :

$$\zeta(x,y;z,a) = \sum_{m=0}^{\infty} \phi(y;z+m,a) \frac{x^m}{m!}, \qquad |y| < 1 \tag{1.5}$$

$$\zeta_v(x,y;z,a) = \sum_{m=0}^{\infty} (v)_m \phi(y;z+m,a) \frac{x^m}{m!}, \qquad |y| < 1, |x| < |a| \tag{1.6}$$

where ϕ is the generalized zeta function.

Its various properties are investigated including the integral representations, generating functions, partial sums and N-fractional calculus. In a recent paper the authors [5] have defined a new generalized zeta function and derived its hypergeometric types of generating functions. In this paper we aim at giving τ-generalizations of the generalized zeta function and at deriving their various properties and formulas including their integral representations, series and genrating functions.

2. Definition and Integral Representations. A τ-generalization of generalized zeta function is defined in terms of series representations as :

$$\phi_\mu^{*\tau}(y;\tau,z,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n \, y^n}{(a+\tau n)^z \, n!}, \tag{2.1}$$

where $|y| < 1, \mathrm{Re}\, a > 0, \mu \geq 1, t \in R, t > 0$.

It is interesting to note that for $\tau = 1$, (2.1) reduces to the generalized zeta function studied by Goyal and Laddha [3].

Here we prove the following integral representation

$$\phi_\mu^{*\tau}(y;\tau,z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at} \left(1 - ye^{-\tau t}\right)^{-\mu} dt, \tag{2.2}$$

where $|y| < 1, \mathrm{Re}\, a > 0, \mu \geq 1, t \in R, t > 0$.

Proof of (2.2). Let

$$I = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at} \left(1 - ye^{-\tau t}\right)^{-\mu} dt$$

$$= \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-at} \sum_{n=0}^{\infty} \frac{(\mu)_n \left(y e^{-\tau t}\right)^n}{n!} dt$$

On interchanging the order of integration and summation, we see that

$$I = \sum_{n=0}^{\infty} \frac{(\mu)_n y^n}{n!(a+\tau n)^z} \frac{1}{\Gamma(z)} \int_0^{\infty} e^{-p} p^{z-1} dp$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_n y^n}{(a+\tau n)^z \, n!}$$

$$= \phi_{\mu}^{*\tau}(y;\tau,z,a).$$

This completes the proof of (2.2).

We now give the τ-generalization of two hypergeometric type generating functions of the Riemann zeta function in the form:

$$e_z^{\tau}(x;\tau) = \sum_{m=0}^{\infty} \zeta^{\tau}(z+m) \frac{x^m}{m!}, \tag{2.3}$$

$|x| < \infty, \tau \in R, \tau > 0,$

and

$$f_z^{\tau}(v;\tau,x) = \sum_{m=0}^{\infty} (v)_m \zeta^{\tau}(z+m) \frac{x^m}{m!}, \tag{2.4}$$

$|x| < 1, \tau \in R, t > 0,$

where v and z are arbitrary fixed complex parameters.

3. Integral Involving $e_z^{\tau}(x;\tau)$ and $f_z^{\tau}(v;\tau,x)$. In this section we evaluate definite integrals involving the function $e_z^{\tau}(x;\tau)$ and $f_z^{\tau}(v;\tau,x)$ in terms of other kind of zeta and hypergeometric functions. First, we recall the Eulerian integral formula of first kind (cf. Srivastava and Manocha [6]);

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \mathrm{Re}(x) > 0, \mathrm{Re}(y) = 0. \tag{3.1}$$

From the term by term integration, we can derive the following

Theorem 1. Let $\mathrm{Re}(c-b) > 0$ and $\mathrm{Re}(b) > 0$. Then

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\, e_z^{\tau}\left(xt^{\tau};\tau\right)dt = G_z^{\tau}(b,c;\tau,x) \tag{3.2}$$

and

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\, f_z^{\tau}\left(\upsilon;\tau,xt^{\tau}\right)dt = G_z^{\tau}(\upsilon,b,c;\tau,x), \tag{3.3}$$

where

$$G_z^{\tau}(b,\mu;c;\tau,x) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\mu)_m\,\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}, \tag{3.4}$$

and

$$G_z^{\tau}(b,c;\tau,x) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}, \tag{3.5}$$

For $\tau = 1$, (3.2) and (3.3) reduce to the known results given earlier by Katsurada [4].

Proof of (3.2). Denote for convenience the left-hand side of relation (3.2) by I. Then in view of (2.3), it is easily seen that

$$I = \sum_{m=0}^{\infty}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b+\tau m-1}(1-t)^{c-b-1}\,dt.$$

Using (3.1), we obtain

$$I = \sum_{m=0}^{\infty}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}\frac{\Gamma(c)\Gamma(b+\tau m)\Gamma(c-b)}{\Gamma(c-b)\Gamma(b)\Gamma(c+\tau m)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}.$$

After some simplification, we get the right-hand side of (3.2).

Proof of (3.3). Denote for convenience the left-hand side of relation (3.3) by J. Then in view of (2.3), it is easily seen that

$$J = \sum_{m=0}^{\infty}\frac{(\upsilon)_m\, x^m}{m!}\zeta^{\tau}(z+\tau m)\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b+\tau m-1}(1-t)^{c-b-1}\,dt.$$

Upon using the Eulerian integral formula (3.1) and the definition (3.4), we are finally led to right-hand side of formula (3.3).

Theorem 2. Let $\operatorname{Re} z > 0$, $\operatorname{Re}\mu > 0$ and $\operatorname{Re}\lambda < 1, \tau \in R, \tau > 0$ Then

$$\frac{1}{\Gamma(\mu)\Gamma(z)}\int_0^\infty\int_0^\infty u^{\mu-1}v^{z-1}e^{-\mu-av}\zeta^\tau\left(xue^{-\tau v},y;\tau,z,a\right)du\,dv$$

$$=\sum_{n=0}^{\infty}\phi_\mu^{*\tau}\left(\frac{x}{(a+\tau n)};\tau,z,a\right)\frac{y^n}{(a+\tau n)^z}, \tag{3.6}$$

and

$$\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}\zeta_v^\tau\left(xe^{\tau t},y;\tau,z,a\right)dt$$

$$=\sum_{n=0}^{\infty}\phi_v^{*\tau}\left(\frac{x}{(a+\tau n)};\tau,z,a\right)\frac{y^n}{(a+\tau n)^z}, \tag{3.7}$$

where

$$\zeta^\tau(x,y;\tau,z,a)=\sum_{m=0}^{\infty}\phi^\tau(y;z+\tau m,a)\frac{x^m}{m!}, \tag{3.8}$$

$|y|<1, \tau\in R, \tau>0, \operatorname{Re}(z)>1, a\neq 0,-1,-2,\ldots$

and

$$\zeta_v^\tau(x,y;\tau,z,a)=\sum_{m=0}^{\infty}(v)_m\,\phi^\tau(y;z+\tau m,a)\frac{x^m}{m!}, \tag{3.9}$$

$|y|<1, \tau\in R, \tau>0, \operatorname{Re}(z)>1, a\neq 0,-1,-2,\ldots$,

while ϕ^τ is the τ-generalized zeta function.

For $\tau=1$, (3.8) and (3.9) reduce to the hypergeometric type generating function studied by Bin-Saad [1].

Proof of (3.6). Denote for convenience the left-hand side of equation (3.6) by L. Then, in view of (3.8), we have

$$L=\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!(a+\tau n)^{z+m}}\frac{1}{\Gamma(\mu)}\int_0^\infty u^{\mu+m-1}e^{-\mu}du\frac{1}{\Gamma(z)}\int_0^\infty e^{-v(a+\tau m)}v^{z-1}dv.$$

Applying the Eulerian integral for gamma function, we get

$$L=\sum_{n=0}^{\infty}\frac{y^n}{(a+\tau n)^z}\sum_{m=0}^{\infty}\frac{(\mu)_m\left[x/(a+\tau n)\right]^m}{m!(a+\tau m)}$$

After some simplification, we obtain the right-hand side of (3.6).

This completes the proof of (3.6). Proof of (3.7) can be developed on the same line.

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SOLUTION OF QUADRATIC DIOPHANTINE EQUATIONS

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ABSTRACT

Our aim is to solve the quadratic Diophantine equations $1161146329226x^2-y^2=\pm 1$. Two starting least positive integer solutions are obtained.

2000 Mathematics Subject Classification : Primary 11GXX, 14GXX; Secondary 11D09.

Keywords: Quadratic Diophantine equation, Quadratic irrational number, Period of Continued fraction, Convergent of continued fraction.

1. Introduction. In Diophantine equations only positive integer solutions are calculated. Solutions of quadratic Diophantine $x^2+19=7^n$ and $x^2+11=3^n$ are discussed by Devi [1]. Unique integer solution of the equation $x^2-13=3^n$ is discussed by Sharma, Singh and Harikishan [3]. Ternary cubic Diophantine equation is solved. Different patterns of non-zero positive integer solutions are obtained by Gopalan and Somnath [2]. Our aim is to find the positive integer solution of quadratic (in two variables) Diophantine equations.

2. Formation of the problem. Consider the quadratic Diophantine equations $Nx^2-y^2=\pm 1$, where N is a positive integer. Let $\sqrt{N}=x_0$ be the quadratic irrational number. The algorithm for $\sqrt{N}$ generates the infinite simple continued fraction of the form $[q_0, q_1, q_2, \ldots, q_n, q_{n+1}, \ldots]$ as follows

$$\left.\begin{array}{l} q_0=[x_0], x_1=\dfrac{1}{x_0-q_0} \\ q_1=[x_1], x_2=\dfrac{1}{x_1-q_1} \\ \ldots \qquad \ldots\ldots \\ q_n=[x_n], x_{n+1}=\dfrac{1}{x_n-q_n} \end{array}\right\}$$

We do not have to calculate infinitely many $q_i's$. Since the quadratic irrational number is always periodic. So it is of the form $\left[q_0, q_1, q_2, \ldots, q_n, \overline{q_{n+1}, \ldots, q_{n+m}}\right]$. It is also found that $q_{n+m} = 2q_0$. Number of terms m, from q_{n+1} to q_{n+m} is called period of continued fraction. If the period is odd both the equations have positive integer solution.

We denote the n^{th} convergent $C_n = P_n / Q_n$, therefore

$$\left.\begin{aligned} C_0 &= \frac{P_0}{Q_0} = \frac{q_0}{1}; \\ C_1 &= \frac{P_1}{Q_1} = \frac{q_0 q_1 + 1}{q_1}; \\ C_n &= \frac{P_n}{Q_n} = \frac{q_n P_{n-1} + P_{n-2}}{q_n Q_{n-1} + Q_{n-2}}, for\ n \geq 2 \end{aligned}\right\} \tag{2.2}$$

The solutions of the equations $Nx^2 - y^2 = \pm 1$ are

$x = Q_1, y = P_i$ for $i = 0, 2, 4, \ldots$ and

$x = Q_i, y = P_i$ for $i = 1, 3, 5, \ldots$. respectively.

3. Results and Discussion.

$\sqrt{1161146329226} = \left[1077565, \overline{2155130}\right]$.

In this case, period of continued fraction is one, which is odd. Therefore both the equations can be solved.

From (2.1) and (2.2), we get

$P_0 = 1077565, Q_0 = 1, P_1 = 2322292658451, Q_1 = 2155130, P_2 = 5004842577008581195$

$Q_2 = 4644585316901, P_3 = 10786086382990825883438801, Q_3 = 100096885154015007260$

Solutions of the equations $1161146329226x^2 - y^2 = \pm 1$ are

x	y
1	1077565
4644585316901	5004842577008581195

and

x	y
2155130	2322292658451
100096885154015007260	10786086382990825883438801

Only two starting least positive integer solutions are calculated.

4. Conclusion. There will be infinite number of solutions. In this problem only two solutions of each of the equation are calculated. One may find the solution of other Diophantine equation of the form $Nx^2 - y^2 = n$, where n is a positive integer.

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q-OPERATIONAL FORMULAE FOR A CLASS OF q-POLYNOMIALS UNIFYING THE GENERALIZED q-HERMITE AND q-LAGUERRE POLYNOMIALS

By

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ABSTRACT

In the present paper, certain q-operational formulae for the generalized q-polynomials $J_n^{(\alpha)}(x;r;p,l)$ are developed and make an attempt to unify the various results. The results given earlier by Gould and Hopper [6], Singh and Srivastava [9], Al-Salam [1], Das [4], Carlitz [3], Joshi and Singhal [10] follow as special cases.

2000 Mathematics Subject Classification : Primary 33D05; Secondary 33D45, 33D50.

Keywords: q-operational formulae, q-Hermite and q-Laguerre polynomials, Fractional q-derivative, q-Gamma function.

1. Introduction. Burchnall [2] made use of the opertional formula

$$(D-2x)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}(x)D^k, \quad D = d/dx, \qquad \text{...(1.1)}$$

to prove the well-known relation

$$H_{m+n}(x) = \sum_{k=0}^{\min[m,n]}(-2)^k\binom{m}{k}\binom{n}{k}K!H_{m-k}(x)H_{n-k}(x). \qquad \text{...(1.2)}$$

Gould and Hopper [6] studied operational formulae associated with classical polynomials and established that

$$\mathcal{G}^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}^r(x,r,p)D^k, \qquad \text{...(1.3)}$$

where the symbol $\mathcal{G}$ is defined by

$$\mathcal{G} = D - prx^{r-1} + \alpha/x$$

and satisfies the relation

$$x^nD^n = \prod_{j=0}^{n-1}\left(xD - prx^r + \alpha - j\right),$$

and

$$H_n^r(x,\alpha,p) = (-1)^n x^{-\alpha} exp(px^r) D^n \left\{x^\alpha \exp(-px^r)\right\} \qquad \text{...(1.4)}$$

defines the elegant generalization of the Hermite polynomials to which it reduces when $\alpha = 0, p = 1, r = 2$.

The relation (1.3) provides a generalization of the formula of Burchnall [2] quoted above as well as of Carlitz's formula [3]

$$\prod_{j=1}^{n}(xD - x + \alpha + j) = n!\sum_{k=0}^{n}\frac{x^k}{k!}L_{n-k}^{(\alpha+k)}(x)D^k, \qquad \text{...(1.5)}$$

for the Laguerre polynomials.

Joshi and Singhal [10] also introduced certain operational formulae associated with a class of polynomials unifying the generalized Hermite and Laguerre polynomials.

The Rodrigues formula (c.f. Joshi and Singhal [10]) is

$$J_n^{(\alpha)}(x;r,p,l) = c(l,n)x^{-\alpha}\exp(px^r)D^n\left\{x^{\alpha+\ln}\exp(-px^r)\right\}, \qquad \text{...(1.6)}$$

where

$$c(l,n) = \frac{(-1)^{n(l-1)(l-2)/2}}{2^{nl(l-1)/2}(1)_{nl(2-l)}},$$

l being a non-negative integer.

In a recent paper, the authors [8] established certain operational formulae for the generalized basic Laguerre polynomials with the help of known results. The object of the present paper is to define q-analogue of the generalized polynomials $J_n^{(\alpha)}(x;r,p,l)$ and develop certain q-operational formulae for the generalized polynomials $J_n^{(\alpha)}(x;r,p,l,q)$ and make an attempt to unify the various results.

For real or complex $\alpha, 0 < |q| < 1$, the q-shifted factorial is defined as

$$(a;q)_n = (q^\alpha;q)_n = \begin{cases}1 & ,n=0\\ (1-q^\alpha)(1-q^{\alpha+1})\ldots(1-q^{\alpha+n-1}) & ,n\in N\end{cases}\cdot \qquad \text{...(1.7)}$$

In terms of the q-gamma function, (1.7) can be expressed as

$$(\alpha;q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, n>0, \qquad \text{...(1.8)}$$

where $\Gamma_q(.)$ is the q-gamma function (c.f. Gasper and Rahman [5]) given by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty (1-q)^{\alpha-1}}.$$

Indeed, it is easy to verify that

$$\lim_{q\to1^-} \Gamma_q(\alpha) = \Gamma(\alpha) \text{ and } \lim_{q\to1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \qquad \text{...(1.10)}$$

where

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1), n \geq 1. \qquad \text{...(1.11)}$$

The fractional q-derivative of arbitrary order $\lambda > 0$ for a function $f(x) = x^{\mu-1}$, is given by

$$D^\lambda_{q,x}\left(x^{\mu-1}\right) = \frac{\Gamma_q(\mu) x^{\mu-\lambda-1}}{\Gamma_q(\mu-\lambda)}, \qquad \text{...(1.12)}$$

where $\mu \neq 0, -1, -2, \ldots$.

For $\lambda = 1$, the equation (1.12) reduces to

$$D_{q,x}\left(x^{\mu-1}\right) = \frac{\Gamma_q(\mu) x^{\mu-2}}{L_q(\mu-1)} = \frac{\left(1-q^{\mu-1}\right) x^{\mu-2}}{(1-q)}. \qquad \text{...(1.13)}$$

Further, we shall denote the infinite product

$$\prod_{j=0}^{\infty} \frac{\left(1-a_1 q^j\right)\ldots\left(1-a_r q^j\right)}{\left(1-b_1 q^j\right)\ldots\left(a-b_s q^j\right)} = \prod\left[\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \end{matrix} q\right] \qquad \text{...(1.14)}$$

In what follows the other notations and symbols employed in this paper have their usual meanings.

2. q-Extension of $J_n^{(\alpha)}(x;r,p,l)$. In this section, we define a q-extension of the generalized polynomials $J_n^{(\alpha)}(x;r,p,l)$ due to Joshi and Singhal [10], by means of the following relation

$$J_n^{(\alpha)}(x;r,p,l,q) = \frac{(-1)^{n(l-1)(l-2)/2}\, x^{-\alpha} e_q\left(px^r\right)}{2^{n(l)(l-1)/2} (l;q)_{nl(2-l)}} D^n_{q,x}\left(x^{\alpha+ln} e_q\left(px^r\right)\right). \qquad \text{...(2.1)}$$

where r,p,l and constants assume integral values, and the q-Leibnitz rule is

$$D_{q,x}^{n}(UV)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}D_{q,x}^{n-r}(U)\,D_{q,x}^{r}(V), \quad \ldots(2.2)$$

where

$$\begin{bmatrix}n\\r\end{bmatrix}=\frac{(q;q)_n}{(q;q)_r\,(q;q)_{n-r}}. \quad \ldots(2.3)$$

By virtue of (1.12) and (1.10), we observe

$$\lim_{q\to 1^-}(1-q)^{n(l-2)}\,J_n^{(\alpha)}(x;r,p,l,q)=J_n^{(\alpha)}(x;r,p,l). \quad \ldots(2.4)$$

3. The q-Operational Formulae. In this section, we prove the various q-operational formulae by working use of the q-differential operator $\delta = xD_q$, which possesses the following interesting properties:

(i) $\quad F(\delta)\left[x^{\alpha}f(x;q)\right]=x^{\alpha}F(\delta+\alpha)f(x;q),$ $\quad\ldots(3.1)$

(ii) $\quad F(\delta)\left[e_q(g(x))f(x;q)\right]=e_q(g(x))F(\delta+xg')f(x;q),$ $\quad\ldots(3.2)$

(iii) $\quad x^{n\alpha}F(\delta)F(\delta+\alpha)\ldots F(\delta+(n-1)\alpha)=\left[x^{\alpha}F(\delta)\right]^{n}.$ $\quad\ldots(3.3)$

We now consider the expression

$$e_q(px^r)x^{-\alpha-ln}D_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right]$$

$$=e_q(px^r)x^{-\alpha-ln-n}x^nD_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right]$$

$$=e_q(px^r)x^{-\alpha-ln-n}\delta(\delta-1)(\delta-2)\ldots(\delta-n+1).\left[x^{\alpha+ln}e_q(-px^r)Y\right] \text{ [By (3.3)]}$$

$$=x^{-n}\prod_{j=1}^{n}(\delta+\alpha+(l-1)n-prx^r+j) \text{ [By making an appeal to (3.1) and (3.2)]}.$$

We thus have

$$D_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right]=x^{\alpha+ln-n}e_q(-px^r)\prod_{j=1}^{n}(\delta+\alpha+(l-1)n-prx^r+j)Y.$$

On the other hand, replacing n by n-k, α by $\alpha+lk$ in (2.1), we find that

$$D_q^n\left[x^{\alpha+ln}e_q\left(-px^r\right)Y\right]=2^{nl(l-1)/2}x^{\alpha}e_q\left(-px^r\right)\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}(1;q)_{(n-k)l(2-l)}$$

$$\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k J_{n-k}^{(\alpha+lk)}(x;r,p,l,q). \qquad ...(3.5)$$

Comparision of (3.4) and (3.5), gives the q-operational formula

$$\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right)=x^{(1-l)n}2^{nl(l-1)/2}\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}$$

$$(1;q)_{(n-k)l(2-l)}\left(\frac{x^l}{2^{l(l-2)/2}}\right)^k J_{n-k}^{(\alpha+lk)}(x;r,p,l,q) \qquad ...(3.6)$$

Also

$$x^{-\alpha}e_q\left(px^r\right)D_q^n\left[x^{\alpha+ln}e_q\left(-px^r\right)Y\right]$$

$$=x^{-\alpha-n}e_q\left(px^r\right)\delta(\delta-1)(\delta-2)...(\delta-n+1)\left[x^{n-k}x^{\alpha+(l-1)n+k}e_q\left(-px^r\right)Y\right].$$

Therefore, making an appeal to (3.1), (3.2), (3.3) and (3.5), we obtain the another q-operational formula

$$x(\delta-k+1)^n\left[x^{\alpha+(l-1)n+k}e_q\left(-px^r\right)Y\right]$$

$$=x^{k+n+\alpha}2^{nl(l-1)/2}e_q\left(-px^r\right)\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}(1;q)_{(n-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{n-k}^{(\alpha+lk)}(x;r,p,l,q).D_q^kY \qquad ...(3.7)$$

Next we observe that the q-analogue of the recurrence relation of Joshi and Singhal [10]:

$$\left(x^lD_q+\alpha x^{l-1}-prx^{l+r-1}\right)J_n^{(\alpha)}(x;r,p,l,q)=\frac{c(l,n)}{c(l,n+1)}J_{n+1}^{(\alpha-1)}(x;r,p,l,q)$$

suggests the q-operational formula

$$\mathcal{J}_{q,l}^{m}J_n^{(\alpha)}(x;r,p,l,q)=\frac{c(l,n)}{c(l,m+n)}J_{m+n}^{(\alpha-nl)}(x;r,p,l,q) \qquad ...(3.8)$$

where

$$\mathcal{J}_{q,L}\equiv x^lD_q+\alpha x^{l-1}-prx^{l+r-1},$$

which corresponds to the q-analogue of the formula of Gould and Hopper [6] to which it reduces when $l=0$.

4. Particular Cases. In this section we shall deduce some interesting particular cases of the q-operational formulae.

(i) If we let $q \to 1^-$ and make use of the limit formula (1.10), in (3.8), we obtain known results due to Gould and Hopper [6].

(ii) If we put $l=0$ and make use of (1.10), our formula (3.6) reduces to (1.3) due to Gould and Hopper [6].

(iii) Again, if we set $p=l=r=1$ in (3.6) and make use of the limit formula (1.10), we arrive at the known results due to Carlitz [3].

(iv) On the other hand, when $l=1$, $q \to 1^-$ and use of (1.10), (3.7) yields the known formula due to Joshi [7].

(v) If we put $l=1$, $p=r=1$ $k=0$, $Y=1$ in (3.7), we get the q-operational formula of Das [4]:

$$\left\{\left(xD_q+1\right)\right\}^n \left\{x^n e_q(-x)\right\} = x^{\alpha+n} e_q(-x)(q;q)_n L_n^{(\alpha)}(x;q).$$

(vi) If we take $l=k=0$, $Y=1$ in (3.7), we obtain

$$\left\{x(xDq+1)\right\}^n \left\{x^{\alpha-n} e_q\left(-px^r\right)\right\} = (-1)^n x^{-\alpha+n} H_n^r(x;r,p,q).$$

5. Applications. Setting $Y=1$ in (3.6), we have

(i) $$\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right)1 = \frac{x^{(1-l)n}}{c(l,n)} J_n^{(\alpha)}(x;r,p,l,q), \quad \text{...(5.1)}$$

so that

$$\frac{x^{(1-l)(m+n)}}{c(l,m+n)} J_{m+n}^{(\alpha)}(x;r,p,l,q)$$

$$= \prod_{j=1}^{m+n}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j\right)1$$

$$= \prod_{j=1}^{m}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j+n\right) \prod_{j=1}^{n}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j\right)1$$

$$= \frac{x^{(1-l)n}}{c(l,n)} \prod_{j=1}^{m}\left(\delta+\alpha+n+(l-1)m-prx^r+j\right). J_n^{\alpha+(l-1)n}(x;r,p,l,q).$$

Therefore in views of (3.6) we finally derive

$$\text{(ii)}\qquad \frac{c(l,m)c(l,n)}{c(l,m+n)}J^{(\alpha)}_{m+n}(x;r,p,l,q)$$

$$=\frac{1}{(1;q)_{ml(2-l)}}\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J^{\alpha+lk+n}_{m-k}(x;r,p,l,q)D_q^k J_n^{\alpha+(l-1)m}(x;r,p,l,q). \qquad ...(5.2)$$

If we reverse the order of the operator on the left hand side in (3.6) and proceed as above,. we obtain an alternative q-operational formula

$$\text{(iii)}\qquad \frac{c(l,m)c(l,n)}{c(l,m+n)}J^{(\alpha)}_{m+n}(x;r,p,l,q)$$

$$=\frac{1}{(1;q)_{ml(2-l)}}\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J^{(\alpha+lk)}_{m-k}(x;r,p,l,q)D_q^k J_n^{(\alpha+lm)}(x;r,p,l,q). \qquad ...(5.3)$$

Comparision of (5.2) and (5.3) gives the identity

$$\text{(iv)}\qquad \sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J^{(\alpha+n+lk)}_{m-k}(x;r,p,l,q)D_q^k J_n^{\alpha+(l-1)m}(x;r,p,l,q)$$

$$=\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J^{(\alpha+lk)}_{m-k}(x;r,p,l,q)D_q^k J_n^{\alpha+lm}(x;r,p,l,q), \qquad ...(5.4)$$

which in the special case $l=0$, making use of (4.6), gives

$$\text{(v)}\qquad \sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}H^r_{m-k}(x;\alpha,p,q)D_q^k H^r_n(x;r,p,q)$$

$$=\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}H^r_{m-k}(x;\alpha+n,p,q)D_q^k H^r_n(x;\alpha q^{-m},p,q) \qquad ...(5.5)$$

for the generalized q-Hermite polynomials.

Next in (5.3) if we replace α by $\alpha - lm$, multiplying on both the sides by

$\dfrac{t^m}{(q;q)_m}$ and summing from $m=0$ to $m=\infty$, we arrive at

$$\text{(vi)} \qquad \sum_{m=0}^{\infty} \frac{(l;q)_{(m+n)l(2-l)}}{(q;q)_m} t^m J_{m+n}^{\alpha-lm}(x;r,p,l,q)$$

$$= (1;q)_{nl(2-l)} J_n^{(\alpha)}\left(x + A_l t x^r, r, p, l, q\right) \sum_{m=0}^{\infty} \frac{(l;q)_{ml(2-l)}}{(q;q)_m} t^m J_m^{(\alpha-lm)}(x;r,p,l,q)$$

where

$$A_l = (-1)^{k(l-1)(l-2)/2} (2)^{-l(l-1)/2} \qquad \ldots(5.6)$$

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ON DISCRETE HERMITE TRANSFORM OF GENERALIZED FUNCTIONS

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ABSTRACT

In this paper discrete Hermite transformation of generalized functions belonging to a certain testing function space has been defined and an inversion formula has been established.

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Keywords: Hermite Transformation, Generalized Functions.

1. Introduction. The aim of the present work is to extend the Hermite transform

$$F(n) = H\{f(x)\} = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) f(x)\,dx \tag{1.1}$$

and its inversion formula

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{F(n)}{2^n n!} H_n(x) (-\infty < x < \infty) \tag{1.2}$$

studied by Debnath [1], to a class of generalized functions. Zemanian [3,4] has also studied (1.1), but here the present work is treated differently.

2. The Notation and Terminology. Throughout this work the interval $(-\infty,\infty)$ is denoted by I. $D(I)$ denotes the space of infinitely differentiable functions on I which have compact support in I. The topology of $D(I)$ is that which makes its dual $D'(I)$ the space of Schwartz distributions. $E(I)$ denotes the space of all infinitely differentiable functions on I. Its dual $E'(I)$ is the space of distributions with compact support. The symobls Ω_x^k denotes the operator

$$\Omega_x^k = \left(D_x e^{-x^2} D_x e^{x^2}\right)^k \equiv \left(D_x^2 + 2xD_x + 2\right)^k,\ D_x \equiv d/dx \tag{2.1}$$

3. Testing Function Space *H(I)* and Its Dual *H'(I)*. We define $H(I)$ as the space of all complex valued infinitely differentiable functions $\phi(x)$ on I such that

$$\rho_k(\phi) = \sup_{-\infty<x<\infty} \left|\Omega_x^k \phi(x)\right| < \infty,\ k = 0,1,2,3,\ldots \tag{3.1}$$

It is seen that $H(I)$ is a linear space. The topology of $H(I)$ is that generated by the

countable multinorm $\{\rho_k\}, k = 0,1,2,3,....$.

$H'(I)$ is the dual of $H(I)$ and is equipped with the usual weak topology. The members of H'(I) are called generalized functions. It can be seen than $H(I)$ is a complete countably multinormed space.

We now give below some properties of the space $H(I)$ and it dual $H'(I)$.

(i) $D(I) \subset H(I)$ and topology of $D(I)$ is stronger than the topology induced on it by $H(I)$. Hence, the restriction of any member of $H'(I)$ to $D(I)$ is in $D'(I)$.

(ii) $D(I) \subset H(I) \subset E(I)$. As $D(I)$ is dense in $E(I)$. $H(I)$ is also a dense subspace of $E(I)$. Consequently, $E'(I)$ can be identified as a subspace of $H'(I)$.

(iii) For each $f \in H'(I)$, there exists a non-negative integar r and a positive constant C such that for all $\phi \in H(I)$.

$$|<f,\phi>| \leq C \max_{0 \leq k \leq r} \rho_k(\phi) \tag{3.2}$$

The proof of this statement follows from the boundedness property of generalized functions.

(iv) If $f(x)$ be a function of x defined in the interval $(-\infty, \infty)$ such that $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then $f(x)$ generates a regular generalized function of $H'(I)$ defined by

$$<f,\phi> = \int_{-\infty}^{\infty} f(x)\phi(x)dx, \phi \in H(I). \tag{3.3}$$

This result can be easily established as follows

$$|<f,\phi>| \leq \rho_0(\phi) \int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

(v) For each positive integer n, the function $\left|e^{-x^2} H_n(x)\right|, (-\infty < x < \infty)$ is a member of $H(I)$. This can be proved as follows :

$$\Omega_x\left[e^{-x^2} H_n(x)\right] = -2n\left\{e^{-x^2} H_n(x)\right\}$$

Hence,

$$\rho_x\left|e^{-x^2} H_n(x)\right| = \sup_{-\infty<x<\infty} \left|\Omega_x^k\left\{e^{-x^2} H_n(x)\right\}\right| = \sup \left|(-2n)^k e^{-x^2} H_n(x)\right| < \infty \forall x, k = 0,1,2,3,.... \tag{3.4}$$

4. The Discrete Hermite Transform of Generalized Functions.

Members of $H'(I)$ are called Discrete Hermite transformable generalized functions.

The Discrete Hermite transform of $f \in H'(I)$ is defined as an application of $f \in H'(I)$ to the kernel $e^{-x^2} H_n(x) \in H(I)$,

i.e. $F(n) = H(f) \equiv \left\langle f(x), e^{-x^2} H_n(x) \right\rangle, n = 0,1,2,\ldots$. (4.1)

As we require time to write the expression $\sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x) H_n(t)}{2^n n!}$, therefore again, we denote it by $T_n(t,x); N$ being any positive integer and $-\infty < x < \infty$, $-\infty < t < \infty$.

We now state below several lemmas whose proofs are based upon the similar lemmas proved in Dube [2].

Lemma 4.1. Let $f \in H'(I)$, then for any positive integer N and for any arbitrary $\phi(x) \in D(I)$

$$\int_{-R}^{R} \left\langle f(t), e^{-t^2} T_N(t,x) \right\rangle e^{-x^2} \phi(x) dx = \left\langle f(t), \int_{-R}^{R} e^{-t^2} T_N(t,x) \right\rangle e^{-x^2} \phi(x) dx \tag{4.2}$$

Lemma 4.2. $\lim_{N \to \infty} \int_{-R}^{R} T_N(t,x) e^{-x^2} dx = 1$.

Lemma 4.3. Let $\phi(x)$ be an arbitrary member of $D(I)$ with compact support in $(-R,R) \in$ I, then

$$\int_{-R}^{R} T_N(t,x) \left[\phi(x) - \phi(t) \right] e^{-x^2} dx \to 0 \text{ as } N \to \infty .$$

5. Inversion Theorem. We now prove the following inversion theorem for our Discrete Hermite transform:

Theorem 5.1. If $F(n)$ denotes the Discrete Hermite transform of $f(t) H'(I)$ as defined in (4.1), then in the sense of convergence in $D'(I)$,

$$f(t) = \lim_{N \to \infty} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} F(n).$$

Proof. Let $\phi(x) \in D(I)$ be an arbitrary member of $D(I)$. We are to prove that

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), \phi(x) \right\rangle \to \ < f(t), \phi(t) > as\ N \to \infty . \tag{5.1}$$

Now $\phi(x) \in D(I) \Leftrightarrow e^{-x^2}, \phi(x) \in D(I)$, hence (5.1) is equivalent prove

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), e^{-x^2} \phi(x) \right\rangle \to \; < f(t), e^{-x^2} \phi(n) > as\ N \to \infty . \tag{5.2}$$

We have

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), e^{-x^2} \phi(x) \right\rangle \tag{5.3}$$

$$= \int_{-R}^{R} \left\{ \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n) \right\} e^{-x^2} \phi(x) dx \tag{5.4}$$

[As $\phi(x) \in D(I)$, the support of $\phi(x)$ is contained in $(-R,R)$].

$$= \int_{-R}^{R} \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} \left\langle f(t), e^{-x^2} H_n(t) \right\rangle e^{-x^2} \phi(x) dx \tag{5.5}$$

$$= \int_{-R}^{R} \left\langle f(t), e^{-t^2} \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x) H_n(t)}{2^n n!} \right\rangle e^{-x^2} \phi(x) dx \tag{5.6}$$

$$= \int_{-R}^{R} \left\langle f(t), e^{-t^2} T_N(t,x) \right\rangle e^{-x^2} \phi(x) dx \tag{5.7}$$

$$= \left\langle f(t), e^{-t^2} \int_{-R}^{R} T_N(t,x) e^{-x^2} \phi(x) dx \right\rangle \tag{5.8}$$

$$\to \left\langle f(t), e^{-t^2} \phi(t) \right\rangle as N \to \infty . \tag{5.9}$$

(5.3) equals to (5.4) as the function $\sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n)$ is locally integrable over the interval I and $\phi(x)$ is in $D(I)$ with support in $(-R,R)$.

(5.6) is obvious because of linearity property of functionals. From Lemma 4.1 we obtain (5.8). Finally, Lemma (4.3) helps us to derive (5.9).

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D_{RS}–TRIANGULATION BASED ON 2^N–RAY ALGORITHM

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ABSTRACT

In order to improve the efficiency of the 2^N-Ray algorithm; we propose a variant of the D_{RS}-Triangulation. A nice property of this triangulation is that it subdivides all the subsets, on which the 2^N-Ray algorithm works, into simplices according to the D_{RS}-Triangulation. Numerical tests shows that 2^N-Ray algorithm based on $D_{1/2}$-Triangulation is much more efficient.

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Keywords: Artificial algorithm, triangulation, admissible lebelling, nodes, barycentre, extra layer.

1. Introduction. Various triangulations have been situdied by Dang [1], Dang and Talman [2], Scarf [4], Todd [6,7], Vander and Talman [8], Vertgeim [9]. The 2^n-ray algorithm was proposed by Wright in [10] to compute solutions of non-linear equations. The 2^n-ray algorithm partitions R^n into $2n$ cones which have the same vertex. Then a triangulation of R^n subdivides each cone into simplices. The 2^n-ray algorithm starts at the vertex and leaves it along an edge of some cone. It follows a sequence of adjacent simplices with varying dimension. Under some mild conditions, the 2^n-ray algorithm terminates at an n-dimensional simplex that yields an approximate solutions to the system of non-linear equations. Since these $2n$ cones have 2^n edges, each of which is a ray, the 2^n–ray algorithm has 2^n possible ray to leave the vertex.

Motivated by this work, we have tried to develop a new D_{RS}–triangulation. The D_{RS}-triangulation based on the 2^n ray algorithm is much more efficient than 2^n ray algorithm proposed by Wright [10] as well as K_1 triangulation introduced by Kuhn in [3] and J_1 triangulation given by Todd [5].

2. Notations and Definitions. The following notations have been used in this paper :

R: Set of real numbers,
Z: Set of all integers,
N: Set of positive integers $(1,2,...,n)$,
N^0: Set of all non-negative integers $N \cup (0)$

R^n : n dimensionally space, having co-ordinates indexed 1 through n,

R^{n+1}: $n+1$ dimensional space, with coordinates indexed 0 through n,

π: Group of permutation on $(1,2,...,n)$ and $\pi+1$ group of permutation on $(0,1,2,...,n)$,

u^i : i^{th} unit vector in R^n, $j \varepsilon N$ and $u = \sum_{i\varepsilon N} u^i$,

R^m_+ : Non negative orthant of $R^m i.e.(x\varepsilon R^m; x \geq 0)$.

Now we consider some standard definitions and explanations which will be used in this paper.

2.1 Standard Simplex. The standard n dimensional closed simplex S^n is the convex hull of $v^0, v^1, ..., v^n i.e. S^n = \{x\varepsilon R^{n+1}_+ : v^T x = 1\}$. s^n_i denotes the face of s^n opposite $v^i i.e. S^n_i = \{x\varepsilon s^n : x_i = 0\}$ and boundary of s^n is denoted by $\partial s = U_{i\varepsilon N} s^n_i$.

Again a j-dimensional simplex or [j-simplex] is the relative interior of the convex hull of $j+1$ affinely independent points $y^0, y^1, y^2, y^3, ..., y^j$, called its vertices. We write $\sigma = \langle y^0, y^1, y^2, ..., y^j \rangle$. A simplex τ is a face of σ if its vertices are a subset vertices of the σ. It is convemient to call the closure of a $(j-1)$ dimensional face of the j simplex σ as a facet of σ. Two j simplices are said to be adjacent if they share a common facet.

2.2 Triangulation. A trangulation G of S^n is a collection of n simplices and satisfies the following two conditions :

1. The simplices in G together with all their faces form a partition of S^n and
2. Each point of S^n has a neighbourhood meeting only a finite number of simplices.

2(a) Pivot Rule. For a given simplices G and a vertex y of σ the rules for obtaining the simplex of G whose vertices include all vertices of σ except y, are called the pivot rules of G.

2(b) Mesh. The mesh of a tringulation G is $\sup_{\sigma\varepsilon G} diam\sigma$. We shall use the Euclidian norm through out this paper.

2.3 Definition. For each sign vector $s\varepsilon R^n$, let

$E(s) = \{ x\varepsilon R^n : s_i x_i = \|x\|$ whenever $s_i \neq 0)$

$= \text{cone}\ \{ t\varepsilon R^n : t$ is a sign vector, and $s_i \neq 0 \Rightarrow s_i = t_i \}$.

In case s has k non-zero components for $k>0$ than $E(s)$ is a polyhedral cone of dimension $n-k+1$. Also we have $E(0) = R^n$. Moreover, when $s \neq 0$ each $E(s) \cap B^n$

is a polyhedral of a cubical subdivisions of B^n where B^n donote the unit ball in 1^∞ norm.

Wright [10] has also defined another subdivision of R^n into closed convex cone as n-dimensional geometric form given as follows :

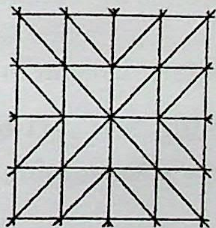
$$Let C(s) = x\varepsilon R^n : \begin{cases} x_i = 0 \text{ if } s_i = 0 \\ s_i x_i \geq 0 \text{ if } s_i \neq 0 \end{cases}$$

$=$cone $\{s_i u_i : s_i 0$, for each sign vector $s\}$. If s has k non-zero components then $C(s)$ is an orthant of a k-dimensional coordinate subspace of R^n.

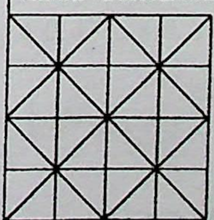
Wright [10] has proposed two type of T-triangulations of R^n with the property that $E(t)$ is a subcomplex for every sign vector $t\varepsilon R^n$ for $t \neq 0$. The first triangulation is called a K^1 triangulation. This triangulation is obtained by taking the triangulation K_1 due to Kuhn [3] in the first orhant and the reflecting through coordinate hyperplane to triangulate the other orhant. A vector v^1 of a n-simplex $<v^1,\ldots,v^{n+1}>$ of K^1 specified by choosing a sign vector s with all the non-zero components, is a member of $C(s)$ when all its components are integrals and π is a permutation of $\{1,2,\ldots,n\}$, then $v^1 \varepsilon K^1$ is defined recursively as :

$V^{i+1} = V^i + S_{\pi(i)} U^{\pi(i)}$, for $i=1,2,\ldots,n$.

For $n=2$ triangulation K^1 can be illustrated by following diagram.



The second triangulation J_1 is defined by Todd [5] and which can be illustrated for $n=2$ by the following figure :



2.3.1. The D_{RS}-*Triangulation*. Define

$$W^n = \{x\varepsilon R^n_+ : x_1 = \max x_i, i = 2,3,\ldots,n\}$$

taking a vector $Y = (y_1, y_2, \ldots, y_n)^T$, we have

$$Y_i = \begin{cases} [x_i] \text{ if } [x_i] \text{ is even} \\ [x_i]+1 \text{ otherwise,} \end{cases}$$

where $[\alpha]$ is the greatest integer less than or equal to α. Let D is the set of all $Y \varepsilon W^n$ where Y_i is defined above. If $Y \varepsilon D$, we define

$I(y) = \{i \varepsilon N; y_1 = y_i\}$ and $J(y) = \{j \varepsilon N : y_1 \geq y_j\}^T$.

Let $s = (s_1, s_2, ..., s_n)^T$ be a sign vector such that

1. For $i \varepsilon N$, if $y_i = 0$ then $s_i = 1$, and if $y_i \neq 0$ then $s_i = -1$.

Let $K(y,s) = \{i \varepsilon I(y) : s_i = 1\}$.

Let ℓ denote the number of element in $I(y)$ and h the number of elements in $K(y,s)$, we take integer p such that

1. when $h=0$, if $\ell = n$ then $p=0$ or 2,
2. when $h>0$, if $h=n$ then $p=0$ and if $h<n$ then $0 \leq p \leq n-1$.

Let $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ be permutation of N.

When $h=0$, for $j=1,2,...,n$,

1. If $j=1$, define

$$g_i(j) = \begin{cases} -1 \text{ if } i \varepsilon I(y) \\ 0 \text{ otherwise} \end{cases} \quad ...(1)$$

for $i=1,2,...,n$.

2. If $j \neq 1$, we define

$$g_i(j) = \begin{cases} S^i \text{ if } i = j, \\ 0 \text{ otherwise} \end{cases} \quad ...(2)$$

for $i=1,2,...,n$.

When $h>0$, for $j=1,2,...,n$.

1. If $\pi(j) \varepsilon K(y,s)$, define

$$g_i(\pi(j)) = \begin{cases} 1 \text{ if } i \varepsilon K(y,s) \text{ and } j \leq \pi^{-1}(i) \\ 0 \text{ otherwise} \end{cases} \quad ...(3)$$

for $i=1,2,...,n$,

2. If $\pi(j) \notin K(y,s)$, define

$$g_i(\pi(j)) = \begin{cases} s_n(j) \text{ if } i = \pi(j) \\ 0 \text{ otherwise} \end{cases} \quad \text{...(4)}$$

for $i = 1,2,\ldots,n$.

If y, π, s and p be as above, then vectors $y^0, y^1, \ldots, y^n$ are defined as follows : for $p = 0$, we have

$$y^0 = y$$

$$y^k = y + g(\pi(k)), k = 1,2,\ldots,n \quad \text{...(5)}$$

and for $p \geq 1$, we define

$$y^0 = y + s,$$

$$y^k = y^{k-1} - s_{\pi(k)} U^{(\pi(k))}, k = 1,2,\ldots,p-1.$$

$$y^k = y + g(\pi(k)), k = p,\ldots,n. \quad \text{...(6)}$$

The $y^0, y^1, \ldots, y^n$ vectors obtained from the above definition are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by $D_{RS}(y,\pi,s,p)$ or $\langle y^0, y^1, \ldots, y^n \rangle$. Let D_{RS} be the set of all such simplices. Then D_{RS} is a triangulation of W^n. Note that simplices of the D_{RS}-triangulation can be represented in more than one way. Moreover, the triangulation of a whole cube in W^n is the same as the D_1-triangulation.

To be more expatiate let us illustrate D_{RS}-triangulation of W^n for $n=2$ and for $x \leq 4$. Obviously, we have that for $y_1 \leq 4$.

$$D = \left\{ (0,0,0)^T, (2,2,0)^T, (4,4,0)^T, (4,0,4)^T, (4,4,4)^T \right\}. \quad \text{...(7)}$$

1. Let $y = (0,0,0)^T$. Then, $I(y) = \{ i \varepsilon N : y_1 = y_i \} = (1,2,3)$ and $\ell = 3$. Then s must be $(1,1,1)^T$. Thus

$K(y,s) = \{ i \varepsilon I(y) : s_1 = 1 \} = (1,2,3)$ and $h = 3$. We have $p = 0$.

(a) Let $\pi = (2,3,1)$. Then $\pi^{-1} = (3,1,2)$ and by applying (3)

$$g(\pi(1)) = g(2) = (g_1(2), g_2(2), g_3(2))$$

$$= (1,1,1)^T,$$

$$g(\pi(2)) = g(3) = (1,0,1)^T,$$

$$g(\pi(3)) = g(3) = (1,0,0)^T.$$

Therefore, $y^0 = y = (0,0,0)^T,$

$$y^1 = y + g(\pi(1)) = (1,1,1)^T,$$

$$y^2 = y + g(\pi(2)) = (1,0,1)^T,$$

$$y^3 = y + g(\pi(3)) = (1,0,0)^T,$$ Let $\sigma^1 = \langle y^0, y^1, y^2, y^3 \rangle$.

(b) Let $\pi = (3,2,1)$. Then $\pi^{-1} = (3,2,1)$

$$g(\pi(1)) = g(3) = (1,1,1)^T,$$

$$g(\pi(2)) = g(2) = (1,1,0)^T,$$

$$g(\pi(3)) = g(1) = (1,0,0)^T,$$

Therefore, $y^0 = y = (0,0,0)^T,$

$$y^1 = y + g(\pi(1)) = (1,1,1)^T,$$

$$y^2 = y + g(\pi(2)) = (1,1,0)^T,$$

$$y^3 = y + g(\pi(3)) = (1,0,0)^T,$$ Let $\sigma^2 = \langle y^0, y^1, y^2, y^3 \rangle$.

2. Let $y = (2,2,0)^T$. Since $I(y) = \{i \varepsilon N : y_1 = y_i\} = \{1,2\}$ and $\ell = 2$.

So s must be $(-1,-1,1)^T$. Thus $k(y,s) = \{i \varepsilon I(y); s_i = 1\} = \phi$ and $h = 0$ while p can be any one of 0,1,2,.

Now by applying (1) and (2), we have

$$g(1) = (-1,-1,0)$$

$$g(2) = (0,-1,0)$$

$$g(3) = (0,0,1)$$

Now considering different values of p and applying (6) and (7), we obtain the following simplices:

(a) For $p=0$. Let $\pi = (1,2,3)$. Therefore,

$$y^0 = y = (2,2,0)^T,$$
$$y^1 = y + g(\pi(1)) = (1,1,0)^T,$$
$$y^2 = y + g(\pi(2)) = (2,1,0)^T,$$
$$y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^3 = \langle y^0, y^1, y^2, y^3 \rangle.$$

(b) For $p=1$. Let $\pi = (1,2,3)$. Therefore

$$y^0 = y + s = (1,1,1)^T,$$
$$y^1 = y + g(\pi(1)) = (1,1,0)^T,$$
$$y^2 = y + g(\pi(2)) = (2,1,0)^T,$$
$$y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^4 = \langle y^0, y^1, y^2, y^3 \rangle.$$

(c) For $p=2$. Let $\pi = (1,2,3)$. We have,

$$y^0 = y + s = (1,1,1)^T,$$
$$y^1 = y^0 - s_{\pi(1)} u^{\pi(1)} = (2,1,1)^T,$$
$$y^2 = y + g(\pi(2)) = (2,1,0)^T,$$
$$y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^5 = \langle y^0, y^1, y^2, y^3 \rangle.$$

3. Let $y = (4,4,0)^T$. Therefore, $I(y) = \{i \varepsilon N; y_1 = y_i\} = \{1,2)$ and $\iota = 2$.

We have that s must be $(-1,-1,1)^T$. Thus $K(y,s) = \{i \varepsilon I(y) : s_i = 1\} = \phi$ and $h=0$.

We have that p can be any one of 0,1,2. We also have

g(1) = (-1,-1,0),
g(2) = (0,-1,0),
g(3) = (0,0,1).

(a) For $p=0$, and $\pi = (1,2,3)$, we have

$$y^0 = y = (4,4,0)^T,$$
$$y^1 = y + g(\pi(1)) = (3,3,0)^T,$$
$$y^2 = y + g(\pi(2)) = (4,3,0)^T,$$
$$y^3 = y + g(\pi(3)) = (4,4,1)^T. \text{ Let } \sigma^6 = \langle y^0, y^1, y^2, y^3 \rangle.$$

(b) For $p=1$, let $\pi=(1,2,3)$. Therefore

$y^0 = y+s=(3,3,1)^T$,

$y^1 = y+g(\pi(1))=(3,3,0)^T$,

$y^2 = y+g(\pi(2))=(4,3,0)^T$,

$y^3 = y+g(\pi(3))=(4,4,1)^T$. Let $\sigma^7=\langle y^0,y^1,y^2,y^3\rangle$.

(c) For $p=2$ and $\pi=(1,2,3)$, we have

$y^0 = y+s=(3,3,1)^T$,

$y^1 = y^0 - s_{\pi(1)}u^{\pi(1)}=(4,4,1)^T$,

$y^2 = y+g(\pi(2))=(4,3,0)^T$,

$y^3 = y+g(\pi(3))=(4,4,1)^T$. Let $\sigma^8=\langle y^0,y^1,y^2,y^3\rangle$.

4. Let $y=(4,0,4)^T$. Since $I(y)=\{i\varepsilon N; y_1 = y_i\}=\{1,3)$ and $\ell=2$. so that s mus

be $(-1,-1,1)^T$. Thus $K(y,s)=\{i\varepsilon I(y): s_i = 1\}=\phi$ and $h=0$. we have that p can be an

one of 0,1,2. We also have

g(1) = (-1,0,-1),

g(2) = (0,1,0),

g(3) = (0,0,-1).

(a) For $p=0$, and $\pi=(1,2,3)$, we have

$y^0 = y=(4,0,4)^T$,

$y^1 = y+g(\pi(1))=(3,0,3)^T$,

$y^2 = y+g(\pi(2))=(4,1,4)^T$,

$y^3 = y+g(\pi(3))=(4,0,3)^T$. Let $\sigma^9=\langle y^0,y^1,y^2,y^3\rangle$

(b) For $p=1$. Let $\pi=(1,2,3)$. Therefore

$y^0 = y+s=(3,1,3)^T$,

$y^1 = y+g(\pi(1))=(3,0,3)^T$,

$y^2 = y+g(\pi(2))=(4,1,4)^T$,

$y^3 = y+g(\pi(3))=(4,0,3)^T$. Let $\sigma^{10}=\langle y^0,y^1,y^2,y^3\rangle$

(c) For $p=2$ and $\pi=(1,2,3)$, we have

$$y^0 = y + s = (3,1,3)^T,$$

$$y^1 = y^0 - s_{\pi(1)} u^{\pi(1)} = (4,1,3)^T,$$

$$y^2 = y + g(\pi(2)) = (4,1,4)^T,$$

$$y^3 = y + g(\pi(3)) = (4,0,3)^T. \text{ Let } \sigma^{11} = \langle y^0, y^1, y^2, y^3 \rangle$$

5. Let $y = (4,4,4)^T$. Since $I(y) = \{i \varepsilon N; y_1 = y_i\} = \{1,2,3)$ and $\ell = 3$. We have that a must be $(-1,-1,-1)^T$. Thus $K(y,s) = \phi$ and $h=0$. we have that p can be any one of 0,1,2. We also have

$g(1) = (-1,-1,-1)$,

$g(2) = (0,-1,0)$,

$g(3) = (0,0,-1)$.

(a) For $p=0$. Let $\pi = (1,2,3)$. Therefore

$$y^0 = y = (4,4,4)^T,$$

$$y^1 = y + g(\pi(1)) = (3,3,3)^T,$$

$$y^2 = y + g(\pi(2)) = (4,3,4)^T,$$

$$y^3 = y + g(\pi(3)) = (4,4,3)^T. \text{ Let } \sigma^{12} = \langle y^0, y^1, y^2, y^3 \rangle.$$

(b) For $p=1$. Let $\pi = (1,2,3)$. Therefore

$$y^0 = y + s = (3,3,3)^T,$$

$$y^1 = y + g(\pi(1)) = (3,3,3)^T,$$

$$y^2 = y + g(\pi(2)) = (4,3,4)^T,$$

$$y^3 = y + g(\pi(3)) = (4,4,3)^T. \text{ Let } \sigma^{13} = \langle y^0, y^1, y^2, y^3 \rangle.$$

(c) For $p=2$. Let $\pi = (1,2,3)$. We have,

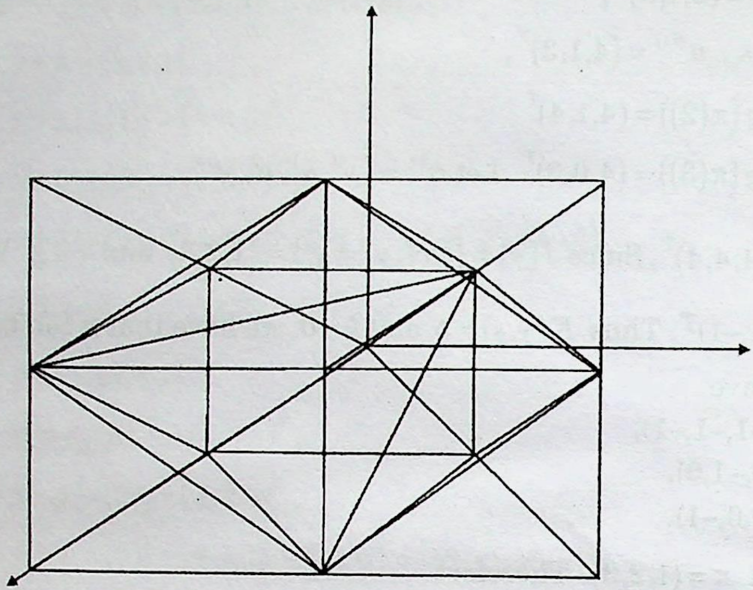
$$y^0 = y + s = (3,3,3)^T,$$

$$y^1 = y^0 - s_{\pi(1)} u^{\pi(1)} = (4,3,3)^T,$$

$$y^2 = y + g(\pi(2)) = (4,3,4)^T,$$

$$y^3 = y + g(\pi(3)) = (4,4,3)^T. \text{ Let } \sigma^{14} = \langle y^0, y^1, y^2, y^3 \rangle$$

It can be seen that $\{\sigma^i : i = 1,2,\ldots,14\}$ form a triangulation of W^3 for $x \le 4$ as shown by the following figure.



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COMPARISON OF A.D.D. OF TRIANGULATIONS

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ABSTRACT

In this paper, making an appeal to computer we try to provide the measure of efficiency for triangulations through graphical represenation.

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Keywords: A.D.D., Triangulations, Fixed Points, Complementary pivoting techniques, Leveling of vertices of the triangulation, Homotopy, Simplex, Sandwich, Finer meshes, Directional density, Theoretical measures of efficiency.

1. Introduction. There are number of algorithms for comuting fixed points using triangulations and complementary pivoting techniques. Each algorithm requires a triangulation, a lebeling of the vertices of the trinagulation, and a particular starting point called as pivoting point.

The successful algorithms use one of the two methods to obtain better and better approximations of fixed points, without wasting previous approximations.

The first one is the restart method of Merril [4], independently developed by Kuhn and Mackinnon [3]. called "sandwich" method. If a fixed point of a mapping f on R^n is sought, one triangulates. $R^n \times [0,1]$ rather than R^n. All vertices lie in either $R^n \times \{0\}$ or $R^n \times \{1\}$. The latter are labelled according to the given mapping f, the former according to the constant map taking each x into $C \varepsilon R^n$. The point c can be so chosen as to be closed to a previous fixed point and provides a starting point for the algorithm.

The second method was introduced by Eaves [1] and extended by Eaves and Saigal [2]. Instead of triangulating S^n or R^n, one triangulates $S^n \times [1,\infty)$ or $R^n \times [1,\infty)$. Roughly, triangulations of finer and finer meshes are placed on the top of the another so that boundary simplices match. Then one generates an infinite path of simplices following fixed point of piecewise linear maps that approximates f better and better as the artificial coordinate tends to ∞. This process of deformation corresponds to the "homotopy" concept. Since we are not concerned

with the maping f, we shall say such triangulations have continuous refinement of grid gize.

The efficiency of fixed point algorithm is very sensitive to the triangulation used. In this paper we are concerned with comparison of the theoretical measures of efficiency for different types of triangulations. The technique of the comparison is based on the concept of directional density of the triangulation in the direction d as defined by Todd [8]. The only other measures known are rather crude measure based on the number of simplices in the unit cube and the diameter, introduced by Saigal, Solow and Wolsey see [7].

In this paper, we have also tried to provide the measure of efficiency for such triangulations through graphical representations by using the computer.

2. Notations and Definitions. The following notations have been used in this paper :

R: Set of real numbers

Z: Set of all integers

E : Set of all even integers.

O: Set of all odd integers

N: Set of all positive integers $\{1,2,...\}$.

N^0: Set of all non-negative inegers $N \cup \{0\}$

R^n : n dimensional space, having co-ordinates indexed 1 through n.

R^{n+1}: $n+1$ dimensional space, with coordinates indexed 0 through n.

π: Group of permutation on $(1,2,...,n)$ and π_{n+1} group of permutation on $(0,1,2,...,n)$

u^i : i^{th} unit vector in R^n, $i \varepsilon N$ and $u = \sum_{i \in N} u^i$ is a vector of ones in R^{n+1}.

V^j: j^{th} unit vector in R^{n+1}, $i \varepsilon N_0$ and $v = \sum_{j \in N} v^j$ is a vector of ones in R^{n+i}.

R^m_+ : Non negative orthant of R^m i.e. $\{x \varepsilon R^m : x \geq 0\}$.

A: Set of $\{-1,+1\}$

B : Set of $\{0,1\}$.

Now we consider some standard definations and explanations which will be used in this paper.

2.1 Simplex. An (open) j-dimensional simplex $\sigma = \langle y^0, y^1, y^2, ..., y^j \rangle$ is the relative interior of the convex hull of $j+1$ affinely independent points $y^0, y^1, ..., y^j$ called its vertices. A simplex τ is a face of σ if its vertices are a subset of vertices of

the σ. Again the closure of a $(j-1)$-dimensional face of the j-simplex σ is called a facet of σ. Two j-simplices are said to be adjacent if they share a common facet.

The diameter of σ is $\sup\{\|x-z\| : x, z\varepsilon\sigma\} = \max\{\|y^i - y^j\| : 0 \le i \le j\}$.

2.2. Triangulation. A triangulation G of an m-dimensional subset F of R^n is a collection of m-simplices that satisfies the following two conditions :

1. The simplices in G, together with all their faces form a partition of F; and
2. Each point of F has a neighbourhood meeting only a finite number of simplices.

We denote G^j the set of j-dimensional faces of simplices in G for $0 \le j \le m$. The members of G^0 is called the vertices of G-and the closures of members of G^{m-1} are called facets of G.

2.3. Pivot Rule. For a given simplex $\sigma\varepsilon G$ and a vertex y of σ, the rules for obtaining the simplex of G whose vertices include all vertices of σ except y are called the pivot rules of G.

2.4. Mesh. The mesh of a triangulation G is sup{diameter of $\sigma : \sigma\varepsilon G$ }. We shall use the euclidean norm throughout this paper.

2.5 Some Standard Regular Traingulations. We define some regular triangulations of R^n. We take the basic grid size of all triangulations to be unity, K,H and J_1 are triangulations of R^n. K (some times called K_1) is based on the standard subdivision of the cube. H is very closely related to K and J_1 is the "*Union Jack*" triangulation of R^n.

All these triangulations use integer vectors as vertices, let $K^0 = H^0 = J_1^0 = Z^n$. Also denote the set of central vertices of J_1 by $J_1^{0c} = \{ y\varepsilon J_1^0 : y_i$ is odd for each $i\varepsilon N$ }.

Now we give definitions of some standard trianglations given as ˋelow :

(1) *K triangulation* is the set of $\sigma = \langle y^0, ..., y^n \rangle$ where $y^0 = y; y^i = y^{i-1} + u^{\pi(i)}$, $1 \le i \le n$

(2) *H triangulation* is the set of $\sigma = \langle y^0, ..., y^n \rangle$ where $y^0 = y; y^i = y^{i-i} + q^{\pi(i)}$,' $1 \le i \le n$

where q^j is the j-th column of the $n \times n$ matrix

$$Q = \begin{bmatrix} -1 & & & 0 \\ +1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & +1 & -1 \end{bmatrix}$$

The simplex o is the set all $x \in R'$ satisfying :

$$1 > -\sum_{1}^{\pi(1)}\left(x_{\pi(1)} - y_{\pi(1)}\right) > \ldots > -\sum_{1}^{\pi(n)}\left(x_{\pi(n)} - y_{\pi(n)}\right) > 0 .$$

(3) J_1 *trinagulation* is the set of $\sigma = \left\langle y^0, \ldots, y^n \right\rangle$ where $y^0 = y; y^i = y^{i-1} + s_{\pi(i)} u^{\pi(i)}$, $1 \le i \le n$ and y^0, j_1^{∞}, π is a permutation of N and $s \varepsilon R^n$ is a sign vector, where $s_i = \pm i$.

The simplex σ is the-set of all $s \varepsilon R^n$ satisfying :

$$1 > s_{\pi(1)}\left(x_{\pi(1)} - Y_{\pi(1)}\right) > \ldots > s_{\pi(n)}\left(x_{\pi(n)} - Y_{\pi(n)}\right) > 0 .$$

The triangulations by J_1 and by K of any unit cube with integral vertices are isomorphic upto orientation of the edges of the cube.

$K_2(m)$ and $J_2(m)$ are the triangulation of S^n having mesh $_{\infty}K_2(m) = m^{-1}$ and $mesh_{\infty} J_2(m) = 2m^{-1}$. Each $K_2(m)$ and $J_2(m)$ can be expressed as follows :

(4) $K_2(m) = \left\{\sigma : \sigma = k_2\left(y^0, \pi\right), \text{ if } \sigma \subseteq S^n\right\}$ is the collection of all such σ where $\sigma = \left\langle y^0, \ldots, y^n \right\rangle$ satisfying $y^i = y^{i-1} + m^{-1} q^{\pi(i)}$ for each $i \varepsilon N$ and where q^j is the jth column of the $(n+1) \times n$ matrix denoted by Q and expressed as

$$Q = \begin{bmatrix} -1 & & & 0 \\ +1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & +1 \end{bmatrix}$$

and its vertices are given by $K_2^0(m) = \left\{y \varepsilon S^n : my_i \varepsilon Z\right\}$ for each $i \varepsilon N_0$.

(5) $J_2(m) = \left\{\sigma : \sigma = j_2\left(y^0, \pi, s\right), \text{ if } \sigma \subseteq S^n\right\}$ is the collection of all such σ where $\sigma = \left\langle y^0, \ldots, y^n \right\rangle$ satisfying $y^i = y^{i-1} + m^{-1} s_{\pi(i)} q^{\pi(i)}$ for each $i \varepsilon N$ and $q^{\pi(i)}$ is as defined in (4) and its vertices are given by $J_2^0(m) = K_2^0(m)$ and centre vertices $J_2^{0c}(m) = \left\{y \varepsilon J_2^0(m) : my_i\right.$ is even for $1 \le i < n$ and my_n is odd$\}$ for each $i \varepsilon N_0$

We now describe two triangulation with continuous refinement of grid size. The first J_3, based on J_1 was introduced by Todd [8], and K_3 is due to Eaves and

Saigal [2].

(6) K_3 triangulates $(0,1]\times R^n$. Let K_3^0 denotes set of vertices $\{y\varepsilon(0,1]\times R^n : y_0 = 2^{-k}$ for $0 \le k\varepsilon Z$ and $y_i / y_0 \varepsilon z$ for $i\varepsilon N\}$. A mapping $t : K_3^{0c} \to A^{n+1}$ is defined by $t_i(y)=0$ or if y_i/y_0 is odd and 1 otherwise. (Thus $t_0(y)$ always equals 0). Let π be a permutation of N_0 and K_3 be the set of $\sigma = k(y,\pi) = \langle y^{-1}, y^0, ..., y^n \rangle$ where

$$y^{-1} = y$$

$$y^i = y^{i-1} + y_0 v^{\pi(i)}, 0 \le i < j = \pi^{-1}(0)$$

$$y^j = y^{j-1} - y_0 \sum_{i=j}^{n} t_{\pi(i)}, v^{\pi(i)} + v^{0-} \sum_{k=j+1}^{n} v^{\pi(k)},$$

$$y^k = y^{k-1} + 2y_0 v^{\pi(k)} = j \le k \le n.$$

The simplex σ is the set of all $x \in R^{n+1}$ satisfying

$$y_0 \ge x_{\pi(0)} - y_{\pi(0)} + t_{\pi(0)}(x_0 - y_0) \ge ... \ge x_{\pi(n)} - Y_{\pi(n)} + t_{\pi(n)}(x_n - y_n) \ge y_0 - x_0.$$

(7) J_3 triangulates $(0,1]\times R^n$. Let J_3^0 denotes set of vertices $\{ y\varepsilon R^{n+1} : y_0 = 2^{-k}$ for $0 \le K\varepsilon Z$ and $y_i / y_0 \varepsilon z$ for $1 \le i \le n \}$.

Let $J_3^{0c} = \{ y\varepsilon J_3^0 : y_i / y_0 \varepsilon O \text{ for } 1 \le i \le n \}$ be the set of central vertices. A mapping $\omega : J_3^{0c} \to A^{n+1}$ is defined by $\omega_i(y) = -1$ or $+1$ according as y_i/y_0 is 1 or 3 mod 4. Let π is a permutation of N_0 and $s\varepsilon A^n$. Then J_3 is the set of simplex $\sigma = j_3(y,\pi,s) = \langle y^{-1}, y^0, ..., y^n \rangle$ where

$$y^{-1} = y,$$

$$y^i = y^{i-1} + y_0 s_{\pi(i)} v^{\pi(i)}, 0 \le i < j = \pi^{-1}(0),$$

$$y^j = y^{j-1} - y_0 \sum_{i=j}^{n} \omega_i(\pi) v^{\pi(i)},$$

$$y^k = y^{k-1} + 2y_0 \omega_\pi(k) v^{\pi(k)}, 1 \le k \le n.$$

The simplex σ is the set of all $x \in R^{n+1}$ satisfying :

$$y_0 \ge s_\pi(0)\left(x_\pi(1) - y_{\pi(1)}\right) \ge ... \ge s_{\pi(j-1)}\left(x_{\pi(j-1)} - y_{\pi(j-1)}\right) \ge x_0 - y_0$$
$$\ge \omega_{\pi(j+1)}\left(x_{\pi(j+1)} - y_{\pi(J+1)}\right) \ge ... \ge \omega_{\pi(n)}\left(x_{\pi(n)} - y_{\pi(n)}\right) \ge y_0 - x_0 .$$

(8) K_3' triangulates $(0,1]\times R^n$. The set of vertices of K_3', denoted by $K_3'^0$ is

precisely J_3^0. We define $\mu : K_3'^0 \to B^{n+1}$ by

$$\mu_i(y) = 1 - y_i / y_0 \bmod 2.$$

Let n is a permutation of N_0 with $\pi(j) = 0$ and $\mu_i(y) = 0$ if $\pi^{-1}(i) > j$ then K_3' is the set of $\sigma = k(y, \pi) = \langle y^{-1}, y^0, \ldots, y^n \rangle$ where

$$y^{-1} = y$$

$$y^i = y^{i-1} + y_0 v^{\pi(i)}, 0 \le i < j$$

$$y^j = y^{j-1} - \sum_{i=0}^{j-1} \mu_i y_0 v^{\pi(i)} + y_0 v^0 - \sum_{k=j+1}^{n} y_0 v^{\pi(k)};$$

$$y^k = y^{k-1} + 2 y_0 v^{\pi(k)}\ j < k \le n.$$

Let $z = x - y$, $\mu = \mu(y)$ and $w = z + z_0 \mu$. Then simplex σ is the set of all $x \varepsilon R^{n+1}$ satisfying :

$$y_0 \ge w_{\pi(0)} \ge \ldots \ge w_{\pi(n)} \ge -w_0.$$

(9) Let A be a non-singular $n \times n$ matrix and G be any triangulation of R^n (here we consider $G = K$ or J_1). Let $AG = (A\sigma : \sigma \varepsilon G\}$ with $A\sigma = \{Ax : x \varepsilon \sigma\}$. Then *trinagulation* AK of R^n is the collection of simplices $\sigma = (y^0, \ldots, y^n)$ such that all components of $A^{-1} y^0$ are integers and $y^i = y^{i-1} + a^{\pi(i)}$ where $a^{(j)}$ is the jth column of A. When A has 1's on the diagonal, -1's on the upper diagonal, and zèroes elsewhere, we have the H triangulation.

(10) J' is a triangulation of R^n defined as follows : The set of vertices of J' is the set of vector $v \varepsilon R^n$ with each component an integer, such that there is not precisely one even component or precisely one odd component. Each simplex $\sigma = j'(y, \pi, s)$ $= (y^0, \ldots, y^n)$, where y is the vector each of the whose components is an even integer, $\pi = (\pi(1), \ldots, \pi(n))$ is a permutation of $(1, 2, \ldots, n)$ and s is a sign vector (each $s_j = \pm 1$) with $s_{\pi(1)} = s_{\pi(n)} = 1$.

For convenience $s_{\pi(i)} e^{\pi(i)}$ is denoted by $\tilde{e}_i$, where $e^{n(i)}$ is the ith unit vector in R^n. For $n \ge 4$ the vertices of j' are defined by

$$y^0 = y$$

$$y^1 = y^0 + 2\tilde{e}^1,$$

$$y^2 = y^1 - \tilde{e}^1 + \tilde{e}^2,$$

$$y^j = y^{j-1} + \tilde{e}^j, 3 \le j \le n-2,$$

$$y^{n-1} = y^{n-2} + \tilde{e}^{n-1} + \tilde{e}^n,$$

$$y^n = y^{n-1} + 2\tilde{e}^n.$$

For $n=3$, each simplices of J' is of the form $\sigma = j'(y,\pi,s) = (y^0, y^1, y^2, y^3)$ where

$$y^0 = y,$$
$$y^1 = y^0 + 2e^{\pi(1)},$$
$$y^2 = y^1 - e^{\pi(i)} + s_{\pi(2)} e^{\pi(2)} + e^{\pi(3)},$$
$$y^3 = y^{2+} e^{\pi(3)}.$$

Now below we give three triangulations of $K._1, J.$ and $J._3$ of $R^n \times [0,1]$ given by Todd [8].

Let $R^n \times [0,1] = \{\bar{x} \varepsilon R^{n+1} : 0 \le \bar{x}^{n+1}\}$. Removal of the bar from a vector in $R^n \times [0,1]$ denotes its projection into R^n. Let $\bar{u}^1, ..., \bar{u}^{n+1}$ be the unit vector in R^{n+1}; thus $u^1, ..., u^n$ are the unit vectors in R^n.

(11) Suppose $\bar{y} \varepsilon Z^n \times \{0\}$ and π is a permutation of $\{1, 2, .., n+1\}$. Then $k_1(\bar{y}, \pi)$ denotes the (closed) simplex $\bar{y}^0, ..., \bar{y}^{n+1}\}$, where

$$\bar{y}^0 = \bar{y},$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{u}^{\pi(i)}, 1 \le i \le n+1.$$

The triangulation K_1 is the set of all such $k_1(\bar{y}, \pi)$'s.

(12) Suppose $\bar{y} \varepsilon Z^n \times \{1\}$ has all $\bar{y}^i$'s odd. Let π be a permutation of $\{1, 2, ..., n+1\}$ and let $\bar{s} \varepsilon R^n \times \{-1\}$ be a sign vector, each $\bar{s}_i$ is ± 1. Then $j_1(\bar{y}, \pi, \bar{s})$ denotes the simplex $[\bar{y}^0, ..., \bar{y}^{n+1}]$, where

$$\bar{y}^0 = \bar{y}$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}_{\pi(i)} \bar{u}^{\pi(i)}, 1 \le i \le n+1\cdot$$

The triangulation J_1 is the set of all such $j_1(\bar{y}, \pi, \bar{s})$'s.

(13) J_3 is the triangulation derived from J_3. Let $\bar{y} \varepsilon Z^n \times \{1\}$ has all $\bar{y}^i$'s odd. Let π be a permutation of $\{1, 2, ..., n+1\}$ with $\pi(j) = n+1$. Let $\bar{s} \varepsilon R^n \times \{-1\}$ be a sign vector such that, for $j \le k \le n+1, \bar{s}n(k)$ is -1 or +1 according as $\bar{y}^{\pi(k)}$ is 1 or 3 mod 4. Then $J_3(\bar{y}, \pi, \bar{s})$ denotes the simplex $[\bar{y}^0, ..., \bar{y}^{n+1}]$, where

$$\bar{y}^0 = \bar{y}$$

$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}\pi(i)\bar{u}^{\pi(i)}, 1 \le i \le j.$$

$$\bar{y}^j = \bar{y}^{j-1} - \sum_{k=j+1}^{n+1} \bar{s}\pi(k)\bar{u}^{\pi(k)} + \bar{s}_{\pi+1}\bar{u}^{n+1},$$

$$\bar{y}^k = \bar{y}^{k-1} + 2\bar{s}\pi(k)\bar{u}^{\pi(k)}, j \le k \le n+1.$$

The triangulation J_3 is the set of all such $J_3(\bar{y},\pi,\bar{s})$'s.

3. Measure of Efficiency for Triangulation. Todd [8] has indicated that the theoretical measure of the efficiency of different types of triangulations used to be compared by counting the number of simplices into which unit cube is divided. However, Todd [8] has also observed that many triangulations have different efficiency but yield the same number of simplices e.g. K, H or J_1 triangulations having $n!$ simplices and K'_3 or J_3 in $\{x\varepsilon R^{n+1} : 0 \le x \le 1, 1/2 \le x_0 \le 1\}$ having $(2^{n+1}-1)\ n!$ simplices.

Saigal Solow and Wolsey introduced the concept of diameter, for those triangulations that subdivided unit cube. This is the maximum over all pairs of facets of the triangulation lying in the facets of the cube as well as of the maximum number of simplices that form a path of simplices linking the two facets. They computed the diameter of K triangulation and an obvious extension to K_3 and found them compareble, Saigal [5] calculated the diameters of triangulations K and H and obtained results and suggesting that the number of iterations using K increases with n^2 while that using H increases with n^3. It is easy to see that the diameters of J_1 and K are equal.

The diameter of triangulation is a "worst best case" measure similar to the diameter of a polytope which is of interest in linear programming.

4. Main Result. First we shall give the concept of average directional density, then formula for obtaining different triangulations and numerical results for obtaining $a.d.d.$ for different values of n. At last we shall give the computerised graph of average directional density of different triangulations.

4.1 Average Directional Density. Todd [8] proposed the concept of directional density as an alternative measure, for the comparison of efficiecy of the triangulations and is global in nature.

For $x, d\varepsilon R^n$ and $\lambda > 0$, let $[x, x+\lambda d]$ denote $\{x+\mu d : \mu\varepsilon[0,\lambda]\}$. Let G be a triangulation of R^n having mesh $\sqrt{n}$. Let $N(G,x,d,\lambda)$ denote the number of simplices of G intersecting $[x, x+\lambda d]$ divided by λ. Let $N(G,x,d,\lambda)$ denote the

limit as $\rho \to \infty$, (if it exists) of the average of $N(G,x,d,\lambda)$ for x uniformly distributed in $B(0,\rho)=\{x\varepsilon R^n : \|x\| < p\}$. Let $N(G,d)$ be the limit as $\lambda \to \infty$ (if it exists) of $N(G,d,\lambda)$. Finally, $N(G)$ is defined to the average of $N(G,d)$ for d uniformly distributed on $\partial B^n = \{d\varepsilon R^n : \|d\| = 1\}$.

We call $N(G,d)$ the directional density of G in direction d and $N(G)$ the average directional density. It should be noted that $N(G,x,d,1)$ is the number of simplices met per unit step size. To eliminate the effect of starting point, we average over x lying in a large ball. To eliminate any effect on the ending point, we let $\lambda \to \infty$. Finally to get a measure independent of the direction d, we average over ∂B^n.

For any triangulation G of R^n, G^{n-2} is a countable collection of sets of dimension n–2. Thus for almost all $x, \{x+\lambda d : \lambda \varepsilon R\}$ meets no simplices of G^+ of dimension less than n–1. If x and $x+\lambda d$ lie in n-simplices of $G.[x,x+\lambda d]$ meet one more simplex of G than (n–1)-simplex of G. Thus the computation of $N(G,x,\lambda,d)$ can be made by counting the number of points of $[x,x+d]$ lying in facet of G, for almost x.

From these observations the following results are obtained by Todd [8] and Vender Laan and Talman [19] :

(i) $N(k_1,d)=\sum_i |d_i| + \sum_{i<j} |d_i - d_j|$

(ii) $N(J_1,d)=\sum_i |d_i| + \sum_{i<j} \frac{1}{2}\left(|d_i + d_j| + |d_i - d_j|\right)$

(iii) $N(H_1,d)=\sum_{i\le j} \left|\sum_{i\le j} d_k\right|$

(iv) $N(J',d)=\sum_{i\le j} \frac{1}{2}\left(|d_i + d_j| + |d_i - d_j|\right)$

(v) Let $g_n = 2\Gamma(n/2)/(n-1)\sqrt{\pi}\Gamma((n-1)/2)$, then

(a) $N(K_1)=N(J_1)=\left(n+\sqrt{2}(n/2)g_n\right)$

(b) $N(H_1)=\sum_1 (n+1-i)\sqrt{(i)}g_n$

(c) $N(K\ )=1/2\sqrt{(n+1)}\sum_i \sqrt{(n+1-i)}\sqrt{(i)}g_n$.

(d) $N(J')=(n/2)\sqrt{2}g_n$

(e) $N\left(\tilde{J}_1\right) = (1/4)\sqrt{(n+1)}\sum_k \sqrt{(n+1-k)}\sqrt{(k)} +$

$$(1/2) \sum_{1\le i\le j\le n} \left(3i+j+1-(i+j+1)\right)/(n+1)\Big)^{1/2}\Big]g_n$$

(f) $N(A*K) = \begin{cases} g_n\{n(n+1)/8\}^{1/2}\{n(n+2)\} \text{ if } n \text{ is even} \\ g_n\{n(n+1)/8\}^{1/2}(n+1) \text{ if } n \text{ is odd} \end{cases}$

Table 1 gives $N(G)/g_n$ for various values of a n and G equal to K, J_1, H , H, A^*K and J'.

TABLE 1

The *a.d.d.* of the K or J_1, K , H, A^*K, J $_1$ and J' triangulations for various values of n(mesh equal to $\sqrt{n}$)

TRIANGULATIONS G

n	K or J_1	K	H	A^*K	$\tilde{J}_1$	J'
1	1	0.7	1	1	0.7	0
2	3.41421	2.44949	3.4142	2.44949	2.698	1.4142
3	7.24264	5.4641	7.5604	4.89898	6.4587	4.2426
4	12.4853	9.94936	13.7047	7.74597	12.098	8.4852
5	19.1421	16.0797	22.089	11.619	20	14.1421
9	59.9117	60.0249	82.0105	33.541	77.6782	50.9116
15	163.492	197.748	269.433	87.6356	261.087	148.4924
20	288.701	392.453	534.189	151.987	524	268.7006
30	645.183	1044.16	1419.95	334.066	1406	615.1829
50	1782.41	3636.58	4942.12	910.357	4336	1732.412
100	7100.36	20108.5	27316.5	3588.52	27462	7000.357

4.2. Asymptotic Behaviour of A.D.D. of Different Triangulations. If G is any triangulation then Asymptotic behaviour of $N(G)/gn$ for $G = K, J_1, K$, $\tilde{J}_1, H$ and AK are given as follows :

(i) For K or J_1 triangulation : $N(K)/g_n =\sim n^2/\sqrt{2} = N(J_1) = N(J')$,

(ii) For K^* triangulation : $N(K^*)/g_n =\sim \pi n^{5/2}/16$,

(iii) For H triangulation : $N(H)/g_n =\sim 4n^{5/2}15$,

(iv) For A^*H triangulation : $N(A*K)/g_n =\sim n^2/\sqrt{8}$,

(v) For $J_1^\bullet$ triangulation : $N(J\bullet_1)/g_n =\sim 0\left(n^{5/2}\right)$.

In table 2 we compared asymptotic behaviour of different triangulations

for different values of n.

TABLE 2

The asymptotic behaviour of *a.d.d.* of the K or J_1, K , H, A^*K and J' triangulations for various values of p (mesh equal to $\sqrt{n}$)

Tringulations G

n	K or J_1	K	H	A^*K	J_1
1	0.70711	0.19643	0.26667	0.35355	1
2	2.82843	1.11117	1.50849	1.41421	5.65685
3	6.36396	3.06202	4.15692	3.18198	15.5885
4	11.3137	6.28571	8.53333	5.65685	32
5	17.6777	10.9807	14.9071	8.83883	55.9017
9	57.2756	47.7321	64.8	28.6378	243
15	159.099	171.172	232.379	79.5495	871.421
20	282.843	351.382	477.028	141.421	1788.85
30	636.396	968.295	1314.53	318.198	4929.5
50	1767.77	3472.4	4714.05	883.883	17677.7
100	7071.07	19642.9	26666.7	3535.53	100000

6. Conclusions. In table 1 it has been observed that *acL.d* of H and J_1 trinagulations increases very fast as n increases and both remains nearly same for n^30. However, their values remain parallel for $30 \leq n \leq 50$. But values for J_1 become faster than H for $50 \leq n \leq 100$. For J' triangulation, in beginning its value increases very slow rate but as n increases its value pick up very fast and become very close to a.d.d. of K or J_1. Rate of increase of *a.d.d.* of A^*K triangulation is the slowest of all the triangulations. In the same way rate for K is faster than K or J_1 but less than H and J_1.

From these observations one can conclude that those triangulations whose *a.d.d.* increases at faster rate are considered to be inferior than the other. On this basis H can be considered inferior among all the triangulations and A^*K is superior among all the triangulations.

Again if we take a graphical representation of tabulated values of a.d.d. for different triangulations, one can easily observed that for $n \leq 15$ there is not much deviation in the different graphs of *a.d.d.* for most of the triangulations. However, between $n=15$ and 20 the deviation between the value of H and K became faster, but that of H and J remain the same, that is H and J_1 begins to deviate at the faster rate than K . Deviation of A^*K is not much from its previous values before $n=15$. In addition to this one can also easily observed that K (or J_1) and J' remain

Chart-I for comparision of a.d.d of the different triangulations for various valus of *n*

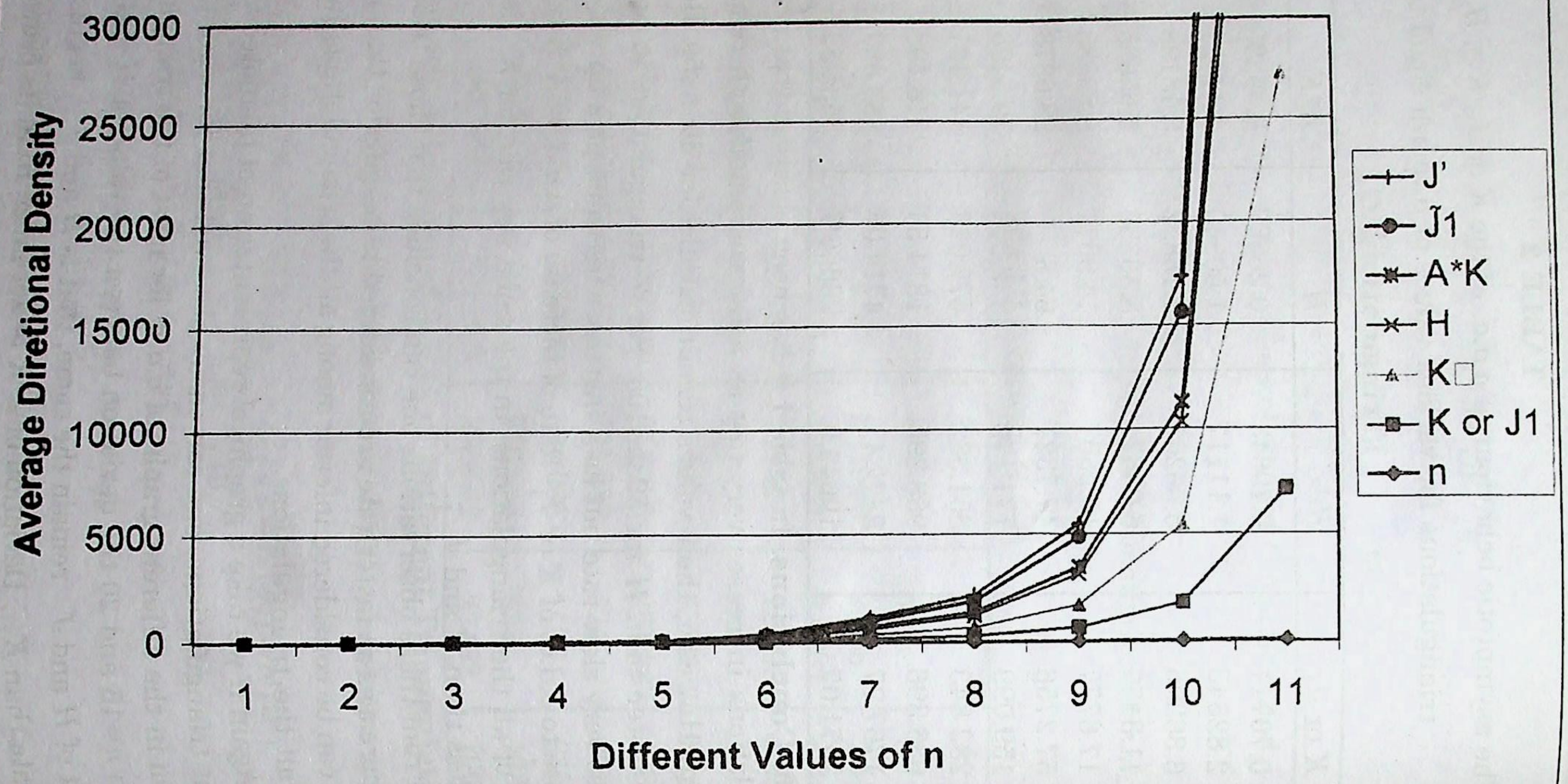
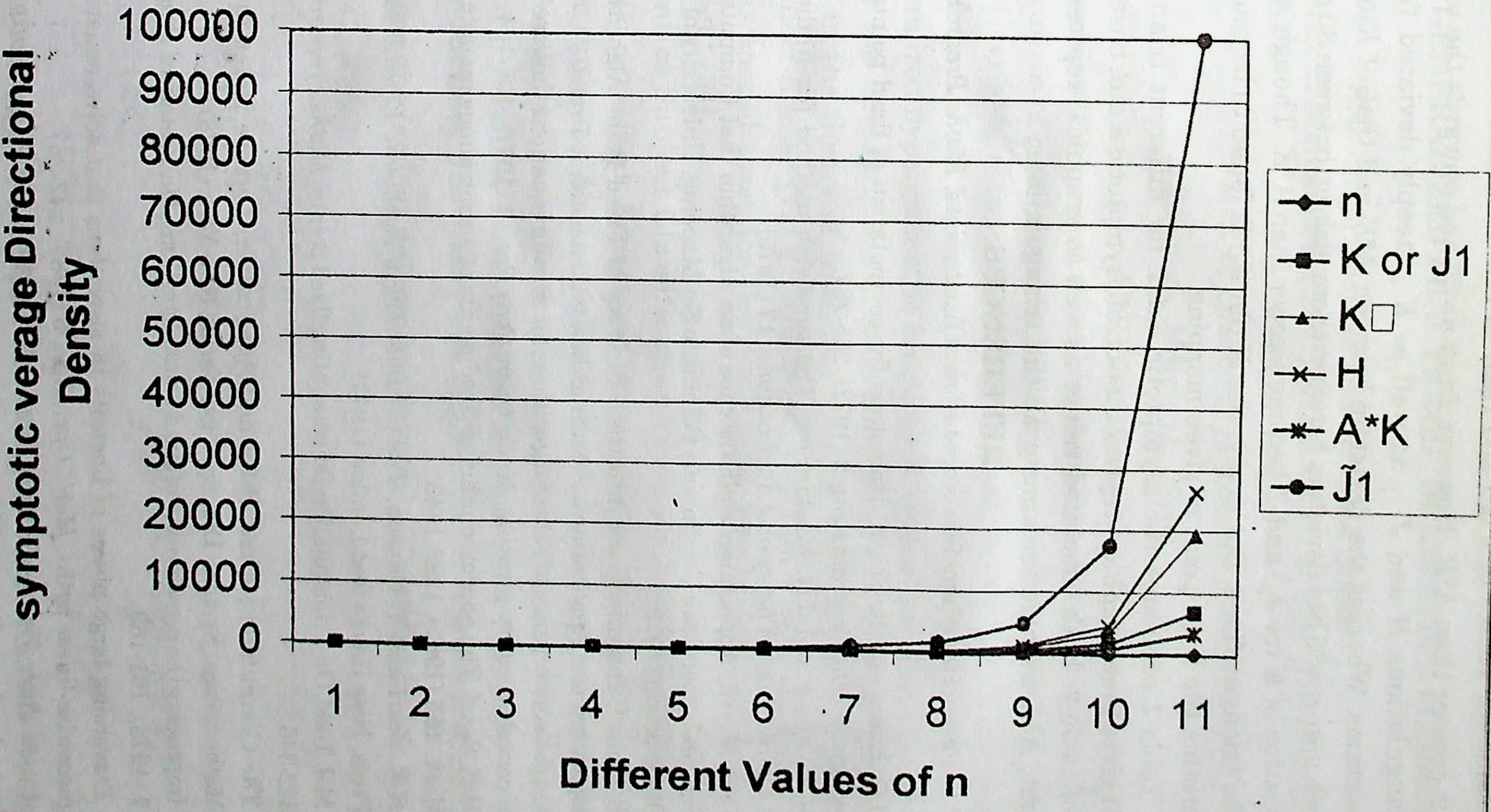


Chart II-comparision of asymptotic a.d.d. of the different triangulations for various values of *n*



nearly same for values of $n \le 20$ and their deviation is also less than H, J_1 and K, but greater than A^*K. However, from $n=30$ and onwards the values of *a.d.d.* for triangulations H and J_1 as well as K steeply deviated from all other triangulations. Whereas the deviation between A^*K and that of K(or J_1) and J' is not much upto $n=30$ but deviation between them pick-up between 30 and 50 thereby the deviation of K (or J_1) and J' became steeper than A^*K. Though A^*K is superior of all the triangulations. But in practice K (for J_1) is found to be much convenient to determine the fixed point of given mapping.

Table-2 compares the asymptotic *a.d.d.* for different triangulations and chart-II gives their graphical representation of asymptotic *a.d.d.* based on tabulated value of table-2. Both tabulated values as well as graphical representation again highlight, A^*K as superior among all the triangulations.

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ON SOME GENERALIZED FRACTIONAL DERIVATIVE FORMULAS

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ABSTRACT

The purpose of the present paper is to derive a number of key formulas for fractional derivatives of generalized multiple hypergeometric functions of several variables, multivariable H-function and genralized multivariable polynomials. Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables. Some of the applications of the new formulas provide potentially useful generalizations of known results in the theory of fractional calculus. Our results include all the results of Chandel-Kumar [12] as special cases and all recent results of Ram-Chandk [32] as special cases for the Fox-Wright generalized hypergeometric function of Wright [54].

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1. Introduction. The theory and applications of fractional calculus are based largely upon the familiar differential operator ${}_{\beta}D_x^{\alpha}$ defined by (cf., e.g., [31, p.49]; see also [47, p.356])

$$(1.1)\qquad {}_{\beta}D_x^{\alpha}\{f(x)\}=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_{\beta}^{x}(x-t)^{-\alpha-1}f(t)dt & (Re(\alpha)<0)\\[2ex] \dfrac{d^m}{dx^m}\,{}_{\beta}D_x^{\alpha-m}\{f(x)\} & (0\le Re(\alpha)<m, m\in N_0)\end{cases}$$

where $N_0=N\cup\{0\}, N=\{1,2,3,...\}$.

For $\beta=0$, equation (1.1) defines the classical Riemann Liouville fractional derivative (or integral) of order $\alpha(or-\alpha)$. On the other hand, when $\beta\to\infty$, the

equation (1.1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order $\alpha(\text{or} - \alpha)$ (see for details, [16, Chap. 13] and [34]).

For the sake of simplicity, the special case of the fractional calculus operator ${}_{\beta}D_x^{\alpha}$ when $\beta = 0$ is written as D_x^{α}. Thus we have

(1.2) $\quad D_x^{\alpha} \equiv {}_0D_x^{\alpha}\,(\alpha \in C).$

For $0 \le \alpha < 1; \beta, \eta, x \in R; m \in N$, the generalized modified fractional derivative operator due to Saigo is defined in Samko, kilbas and Marichev [34] as

(1.3) $$D^{\alpha,\beta,\eta}_{0,x,m} f(x) = \frac{d}{dz}\left(\frac{z^{-m(\beta-\eta)}}{\Gamma(1-\alpha)}\int_0^x \left(x^m - t^m\right)^{-\alpha} F(\beta-\alpha),1-\eta;1-\alpha;1-t^m/x^m\right) f(t)\,dt^m.$$

The Multiplicity of $x^m - t^m$ in above equation is removed by requiring $log\left(x^m - t^m\right)$ as real for $x^m - t^m > 0$ and is assumed to be well defined in the unit disk.

It is remarkable that

(1.4) $\quad D^{\alpha,\alpha,\eta}_{0,x,1} f(x) = D_x^{\alpha} f(x),$

where D_x^{α} is the familiar Riemann-Liouville fractional derivative operator defined by Miller and Ross [28]

For $0 \le \alpha < 1, m \in N; \beta, \eta, x \in R;\ \mu > max(0, \beta - \eta)$, the refined form due to Bhatt and Raina [2] is given by

(1.5) $$D^{\alpha,\beta,\eta}_{0,x,m}\left\{x^{(\mu-1)m}\right\} = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, x^{(\mu-\beta-1)m}.$$

Making an appeal to the Saigo modified fractional derivative operator $D^{\alpha,\beta,\eta}_{0,x,m}$, Miller and Rasso [28] investigated fractional derivative formulas. Kilbas [21] established the fractional integral formulas for the Wright ([54]; see also Erdélyi [14]) introduced the generalized hypergeometric function ${}_p\Psi_q$ Fox-Wright generalized hypergeometric function, defined by

(1.6) $${}_p\Psi_q(z) = {}_p\Psi_q\left[\left.\begin{matrix}(a_i, A_i)_{1,p}\\ (b_j, B_j)_{1,q}\end{matrix}\right| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i k)}{\prod_{j=1}^{q}\Gamma(b_j + B_j k)} \frac{z^k}{k!}$$

where $a_i, b_j \in C; A_i > 0, B_j > 0; 1+\sum_{j=1}^{q} - \sum_{i=1}^{p} A_i \geq 0;\ (A_i, B_j \neq 0)(i=1,\ldots,p; j=1,\ldots,q)$, for suitable bounded values of $|z|$.

For details of conditions of its existence and the H-function due to Mathai and Saxena [26], see Kilbas [21].

Wright [55] also introduced the special case of (1.6) (called Wright function) defined in the form:

$$(1.7)\quad \phi(\alpha,\beta;z) = {}_0\Psi_1\left[\left.\begin{matrix} \\ (\alpha,\beta)\end{matrix}\right| z\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta k+\alpha)} \frac{z^k}{k!},$$

where $\alpha, z \in C$ and $\beta \in R$.

Kiryakova [23] introduced a function $J_{\nu}^{\delta}(z)$ called Bessel-Maitland function or Wright generalized Bessel function defined by

$$(1.8)\quad J_{\nu}^{\delta}(z) = \phi(\upsilon+1,\delta;-z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k+\nu+1)} \frac{(-z)^k}{k!}.$$

Other special case of (1.4) but generalizing the classical Mittag-Leffler function (Erdélyi [16]) is given by Kilbas et al. [22].

Srivastava and Garg [38] introduced a general class of multivariable polynomials defined by

$$(1.9)\quad S_L^{h_1,\ldots,h_s}(x_1,\ldots,x_s) = \sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s \leq L} (-L)_{h_1k_1+\ldots+h_sk_s} A(L;k_1,\ldots,k_s) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!}$$

where $h_1,\ldots,h_s$ are arbitrary positive integers and the coefficients $A(L;k_1,\ldots,k_s)$, $(L;k_i \in N_0; i=1,\ldots,s)$ are arbitrary constants real or complex, $N_0 = N \cup \{0\}$.

It is clear that for $s=1$, the polynomials (1.9) reduce to the polynomials of Srivsatava [37] defined by

$$(1.10)\quad S_L^h(x) = \sum_{k=0}^{[l/h]} \frac{(-l)_{hk}}{k!} A_{l,k} x^k \qquad (l \in N_0 = \{0,1,2,\ldots\}),$$

where h is arbitrary positive integer and the coefficients $A_{l,k}(l;k \in N_0)$ are arbitrary constants, real or complex.

The computation of fractional derivatives (and fractional integrals) of

special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf., e.g., [29] and [52]), the derivative of generating functions [46, Chap. 5] and solutions of differential and integral equation (cf. [29] and [39], Chap. 3. see also [27, 30 and 51]. Making an appeal to the operator (1.4), Chandel and Vishwakarma ([6],[7]) have obtained fractional derivatives of confluent forms due to Chandel-Vishwakarma [5]) of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ [20] and other multiple hypergeometric functions of Lauricella [24], Chandel [3], Chandel and Gupta [4] including their confluent forms. Srivastava and Goyal [43] derived fractional derivatives of the multivariable H-function of Srivastava and Panda ([48]-[50]). Srivastava, Chandel and Vishwakarma [40] obtained several new fractional derivative formulas involving the multivariable H-function defined by Srivastava and Panda (see [50, p.271, eq. [4.1] et Seq.]) and studied systematically by them (see [48] [50] also [44]).

Further for special interest, Chandel and Vishwakarma [8] and Chandel-Sharma [11] derived fractional derivatives involving hypergeometric functions of four variables defined by Exton [18] and Sharma-Parihar [35], while Chandel-Sharma [11] established fractional derivative formulas for their own hypergeometric functions of four variables ([9],[10]). Recently, Chandel and Kumar [12] derived generalizations and unifications of various key formulas of Srivatava Chandel and Vishwakarma [40]. Very recently employing the operator (1.3), Ram and Chandak [32] derived a generalized derivative formula involving the product of Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ defined by (1.6) and a general class of multivariable polynomials defined by (1.9). Some special cases are also discussed.

In the present paper, employing the operator (1.3), with the motivation of Chandel-Kumar [12] and Ram-Chandak [32], we derive generalizations and unifications of various key formulas of Chandel-Kumar [12] and Ram-Chandak [32]. Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables and genearlized hypergeometric functions of one variable. Some of applications of the key formulas provide potentially useful genearlizations of known results in the theory of fractional calculus. Some special cases are also discussed.

2. Key Formulas. In this section, making an appeal to the result (1.5), we derive the following key formulas on generalized fractional derivatives involving multiple hypergeometric function of Srivastava-Daoust ([41],[42]; also see Srivastava-Manocha [46, p.64 (18), (19), (20)]); multivariable H-function of

Srivsatava-Panda ([48]-[50]; see also Srivastava, Gupta-Goyal [44]), generalized multivariable polynomials due to Srivastava-Garg [38]:

$$(2.1)\quad D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\Big\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}$$

$$F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c);\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.$$

$$\left.z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\dots x_r^{m_r\rho'_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma'_r},\dots,z_nx_1^{m_1\rho''_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma''_1}\dots x_r^{m_r\rho''_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma''_r}\right)\Big\}$$

$$=\prod_{j=1}^{r}\frac{\Gamma(\mu_j)\Gamma(\mu_j+\eta_j-\beta_j)}{\Gamma(\mu_j-\beta_j)\Gamma(\mu_j+\eta_j-\alpha_j)}x_j^{(\mu_j-\beta_j-1)m_j}\xi_j^{\lambda_j}F_{C+3r:D';\dots;D^{(n)};0;\dots;0}^{A+3r:B';\dots;B^{(n)};0;\dots;0}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)},0,\dots,0\right],\\ \left[(c):\psi',\dots,\psi^{(n)},0,\dots,0\right],\end{matrix}\right.$$

$$\begin{matrix}\left[\mu_1:\rho'_1,\dots,\rho''_1,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\beta_1:\rho'_1,\dots,\rho''_1,\nu_1,0,\dots,0\right],\dots,\left[\mu_r:\rho'_r,\dots,\rho''_r,0,\dots,0,\nu_r\right],\\ \left[\mu_1-\beta_1:\rho'_1,\dots,\rho''_1,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\alpha_1:\rho'_1,\dots,\rho''_1,\nu_1,0,\dots,0\right],\dots,\left[\mu_r-\beta_r:\rho'_r,\dots,\rho''_r,0,\dots,0,\nu_r\right]\end{matrix}$$

$$\begin{matrix}\left[\mu_r+\eta_r-\beta_r:\rho'_r,\dots,\rho''_r,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma''_1,1,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma''_r,0,\dots,0,1\right]:\\ \left[\mu_r-\eta_r-\alpha_r:\rho'_r,\dots,\rho''_r,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma''_1,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma''_r,0,\dots,0\right]:\end{matrix}$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];-;\dots;-;\\ \left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];-;\dots;-;\end{matrix}\;Z_1,\dots,Z_n,\frac{-x_1^{m_1}}{\xi_1},\dots,\frac{-x_r^{m_r}}{\xi_r}\right),$$

where $F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([41],[42]), $0\le\alpha_i<1;m_i\in N;\beta_i,\eta_i x_i\in R;\mu_i>max(0,\beta_i-\eta_i)$,

$$Z_i=\frac{z_ix_1^{\rho_1^{i}m_1}\dots x_r^{\rho_r^{i}m_r}}{\xi_1^{\sigma_1^{i}}\dots\xi_r^{\sigma_r^{i}}},\ i=1,\dots,n;$$

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\prod_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,\ i=1,\dots,n$$

and

$$\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0 \quad i=n+1,\dots,n+r.$$

(2.2) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}\right.$$

$$H_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}\left(\begin{matrix}[(a),\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^n):\phi^n];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.$$

$$\left.\left.z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\dots x_r^{m_r\rho'_r}\left(x_r^{m_r\nu_r}+\xi_n\right)^{-\sigma'_r},\dots,z_nx_1^{m_1\rho''_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma''_1}\dots x_r^{m_r\sigma''_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma''_r}\right)\right\}$$

$$=\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}x_1^{(\mu_1-\beta_1-1)m_1}\dots x_r^{(\mu_r-\beta_r-1)m_r}\cdot\sum_{N_1,\dots,N_r=0}^{\infty}\frac{\left(-x_1^{\nu_1m_1}/\xi_1\right)^{N_1}}{N_1!}\dots\frac{\left(-x_r^{\nu_rm_r}/\xi_r\right)^{N_r}}{N_r!}$$

$$H_{A+3r,C+3r:[B',D'];\dots;[B^{(n)},D^{(n)}]}^{0,\lambda+3r:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[1-\mu_1-\nu_1N_1:\rho'_1,\dots,\rho''_1],\\ [(c):\psi',\dots,\psi^{(n)}],[1-\mu_1-\nu_1N_1+\beta_1:\rho'_1,\dots,\rho''_1],\end{matrix}\right.$$

$$\begin{matrix}[1-\mu_1-\eta_1+\beta_1-\nu_1N_1:\rho'_1,\dots,\rho''_1],[1+\lambda_1-N_1:\sigma'_1,\dots,\sigma''_1],\dots,[1-\mu_r-\nu_rN_r:\rho'_r,\dots,\rho''_r],\\ [1-\mu_1-\nu_1N_1-\eta_1+\alpha_1:\rho'_1,\dots,\rho''_1],[1+\lambda_1:\sigma'_1,\dots,\sigma''_1],\dots,[1-\mu_r-\nu_rN_r+\beta_r:\rho'_r,\dots,\rho''_r],\end{matrix}$$

$$\begin{matrix}[1-\mu_r-\eta_r+\beta_r-\nu_rN_r:\rho'_r,\dots,\rho''_r],[1+\lambda_r-N_r:\sigma'_r,\dots,\sigma''_r]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [1-\mu_r-\eta_r+\alpha_r-\nu_rN_r:\rho'_r,\dots,\rho''_r],[1+\lambda_r:\sigma'_r,\dots,\sigma''_r]:\quad[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}$$

$$\left.Z_1,\dots,Z_n\right)$$

provided that $0\le\alpha_i<1, m_i\in N;\beta_i\eta_i,x_i\in R;\mu_i>Max(0,\beta_i-\eta_i)$,

$max\left\{arg\left(x_1^{\nu_1m_1}/\xi_1\right),\dots,arg\left(x_r^{\nu_rm_r}/\xi_r\right)\right\}<\pi, min\left(\nu_1,\dots,\nu_r;\rho_1^i,\dots,\rho_r^i;\sigma_1^i,\dots,\sigma_r^i\right)>0$,

$Z_i=\dfrac{z_ix_1^{\rho_1^im_1}\dots x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}\dots\xi_r^{\sigma_r^i}}$, $i=1,\dots,n$ and where $H_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}$ is multivariable H-

function due to Srivastava and Panda ([48]-[50]).

(2.3) $$D^{\alpha_1,\beta_1,\eta_1}_{0,x_1,m_1}\ldots D^{\alpha_r,\beta_r,\eta_r}_{0,x_r,m_r}\left\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\ldots x_r^{(\mu_r-1)m_r}\left(x_r^{x_r\nu_r}+\xi_r\right)^{\lambda_r}\right.$$

$$F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{n}\right):\delta^{(n)}\right];\end{matrix}\ z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\right.$$

$$\left.\ldots x_r^{m_r}\rho'_r\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma'_r},\ldots,z_nx_1^{m_1\rho''_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma''_1}\ldots x_r^{m_r}\rho''_r\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma''_r}\right)$$

$$F^{E:F';\ldots;F^{(s)}}_{G:H';\ldots;H^{(s)}}\left(\begin{matrix}\left[(e):\alpha',\ldots,\alpha^{(s)}\right]:\left[(f'):\beta'\right];\ldots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];\\ \left[(g):\gamma',\ldots,\gamma^{(s)}\right]:\left[(h'):\eta'_1\right];\ldots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];\end{matrix}\ w_1x_1^{k'_1m_1}\ldots x_r^{k'_rm_r},\ldots,\right.$$

$$\left.\left.w_sx_1^{k^s_1m_1}\ldots x_r^{k^s_rm_r}\right)\right\}$$

$$=\prod_{i=1}^{r}\frac{\xi_i^{\lambda_i}\Gamma(\mu_i)\Gamma(\mu_i+\eta_i-\beta_i)}{\Gamma(\mu_i-\beta_i)\Gamma(\mu_i+\eta_i-\alpha_i)}x_i^{(\mu_i-\beta_i-1)m_i}$$

$$F^{A+E+3r:\beta';\ldots;\beta^{(n)};F';\ldots;F^{(s)};0;\ldots;0}_{C+G+3r:D';\ldots;D^{(n)};H';\ldots;H^{(s)};0;\ldots;0}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)},0,\ldots,0\right],\left[(e):0,\ldots,0,\alpha',\ldots,\alpha^{(s)},0,\ldots,0\right],\\ \left[(c):\psi',\ldots,\psi^{(n)},0,\ldots,0\right],\left[(g):0,\ldots,0,\gamma',\ldots,\gamma^{(s)},0,\ldots,0\right],\end{matrix}\right.$$

$$\begin{matrix}\left[\mu_1;\rho'_1,\ldots,\rho''_1,k'_1,\ldots,k^s_1,\nu_1,0,\ldots,0\right],\left[\mu_1+\eta_1-\beta_1:\rho'_1,\ldots,\rho''_1,k'_1,\ldots,k^s_1,\nu_1,0,\ldots,0\right],\ldots,\\ \left[\mu_1-\beta_1:\rho'_1,\ldots,\rho''_1,k'_1,\ldots,k^s_1,\nu_1,0,\ldots,0\right],\left[\mu_1+\eta_1-\alpha_1:\rho'_1,\ldots,\rho''_1,k'_1,\ldots,k^s_1,\nu_1,0,\ldots,0\right],\ldots,\end{matrix}$$

$$\begin{matrix}\left[\mu_r;\rho'_r,\ldots,\rho''_r,k'_r,\ldots,k^s_r,0,\ldots,0,\nu_r\right],\left[\mu_r+\eta_r-\beta_r:\rho'_r,\ldots,\rho''_r,k'_r,\ldots,k^s_r,0,\ldots,0,\nu_r\right],\\ \left[\mu_r-\beta_r:\rho'_r,\ldots,\rho^n_r,k'_r,\ldots,k^s_r,0,\ldots,0,\nu_r\right],\left[\mu_r+\eta_r-\alpha_r:\rho'_r,\ldots,\rho^n_r,k'_r,\ldots,k^s_r,0,\ldots,0,\nu_r\right],\end{matrix}$$

$$\begin{matrix}\left[-\lambda_1:\sigma'_1,\ldots,\sigma''_1,0,\ldots,0,1,0,\ldots,0\right];\ldots;\left[-\lambda_r:\sigma'_r,\ldots,\sigma''_r,0,\ldots,0,1\right]:\\ \left[-\lambda_1:\sigma'_1,\ldots,\sigma^n_1,0,\ldots,0\right],\ldots,\left[-\lambda_r:\sigma'_r,\ldots,\sigma''_r,0,\ldots,0\right]:\end{matrix}$$

$$\begin{matrix}\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\left[(f'):\beta'\right];\ldots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];-;\ldots;-;\\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\left[(h'):\eta'\right];\ldots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];-;\ldots;-;\end{matrix}\ Z_1,\ldots,Z_n,$$

$$w_1 x_1^{k'_1 m_1} \ldots x_r^{k'_r m_r}, \ldots, w_r x_1^{k_1^s m_1} \ldots x_r^{k_r^s m_r}, -x_1^{\nu_1 m_1}/\xi_1, \ldots, -x_r^{\nu_r m_r}/\xi_r \Big),$$

valid if $Z_i = \dfrac{z_i x_1^{\rho_1^i m_1} \ldots x_r^{\rho_r^i m_r}}{\xi_1^{\sigma_1^i} \ldots \xi_r^{\sigma_r^i}}$, $0 \le \alpha_i < 1, m_i \in N; \beta_i, \eta_i, x_i \in R; \mu_i > Max(0, \beta_i - \eta_i)$,

$min(\nu_1, \ldots, \nu_r; \rho_1^i, \ldots, \rho_r^i; \sigma_1^i; \ldots, \sigma_r^i) > 0, i = 1, \ldots, n.$

(2.4) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \Big\{ x_1^{(\mu_1-1)m_1} \left(x_1^{m_1\nu_1} + \xi_1\right)^{\lambda_1} \ldots x_r^{(\mu_r-1)m_r} \left(x_r^{m_r\nu_r} + \xi_r\right)^{\lambda_r}$$

$$F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^n):\delta^{(n)}]; \end{matrix} z_1 x_1^{m_1\rho'_1} \left(x_1^{m_1\nu_1} + \xi_1\right)^{-\sigma'_1} \right.$$

$$\left. \ldots x_r^{m_r\rho'_r} \left(x_r^{m_r\nu_r} + \xi_r\right)^{-\sigma'_r}, \ldots, z_n x_1^{m_1\rho_1^n} \left(x_1^{m_1\nu_1} + \xi_1\right)^{-\sigma_1^n} \ldots x_r^{m_r\rho_r^n} \left(x_r^{m_r\nu_r} + \xi_r\right)^{-\sigma_r^n} \right)$$

$$S_L^{h_1,\ldots,h_s} \left(w_1 x_1^{k'_1 m_1} \ldots x_r^{k'_r m_r}, \ldots, w_s x_1^{k_1^s m_1} \ldots x_r^{k_r^s m_r} \right) \Big\}$$

$$= \frac{\xi_1^{\lambda_1} \ldots \xi_r^{\lambda_r} \Gamma(\mu_1)\Gamma(\mu_1+\eta_1-\beta_1) \ldots \Gamma(\mu_r)\Gamma(\mu_r+\eta_r-\beta_r)}{\Gamma(\mu_1-\beta_1)\Gamma(\mu_1+\eta_1-\alpha_1) \ldots \Gamma(\mu_r-\beta_r)\Gamma(\mu_r+\eta_r-\alpha_r)} x_1^{(\mu_1-\beta_1-1)m_1} \ldots x_r^{(\mu_r-\beta_r-1)m_r}$$

$$\sum_{R_1,\ldots,R_s=0}^{h_1R_1+\ldots+h_sR_s \le L} (-L, h_1R_1 + \ldots + h_sR_s) A(L:R_1,\ldots,R_s)$$

$$\frac{(\mu_1, k'_1 R_1 + \ldots + k_1^s R_s) \ldots (\mu_r, k'_r R_1 + \ldots + k_r^s R_s)}{(\mu_1 - \beta_1, k'_1 R_1 + \ldots + k_1^s R_s) \ldots (\mu_r - \beta_r, k'_r R_1 + \ldots + k_r^s R_s)}$$

$$\frac{(\mu_1 + \eta_1 - \beta_1, k'_1 R_1 + \ldots + k_1^s R_s) \ldots (\mu_r + \eta_r - \beta_r, k'_r R_1 + \ldots + k_r^s R_s)}{(\mu_1 + \eta_1 - \alpha_1, k'_1 R_1 + \ldots + k_1^s R_s) \ldots (\mu_r + \eta_r - \alpha_r, k'_r R_1 + \ldots + k_r^s r_s)}$$

$$\frac{\left(w_1 x_1^{k'_1 m_1} \ldots x_r^{k'_r m_r}\right)^{R_1}}{R_1!} \ldots \frac{\left(w_s x_1^{k_1^s m_1} \ldots x_r^{k_r^s m_r}\right)^{R_s}}{R_s!} F_{C+3r:D';\ldots;D^{(n)};0;\ldots;0}^{A+3r:B';\ldots;B^{(n)};0;\ldots;0} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)},0,\ldots,0], \\ [(c):\psi',\ldots,\psi^{(n)},0,\ldots,0], \end{matrix} \right.$$

$$[-\lambda_1 : \sigma'_1, \ldots, \sigma_1^n, \nu_1, 0, \ldots, 0], \ldots, [-\lambda_r : \sigma'_r, \ldots, \sigma_r^n, 0, \ldots, 0, \nu_r],$$
$$[-\lambda_1 : \sigma'_1, \ldots, \sigma_1^n, 0, \ldots, 0], \ldots, [-\lambda_r : \sigma'_r, \ldots, \sigma_r^n, 0, \ldots, 0],$$

$$\left[\mu_1+k'_1 R_1+\ldots+k_1^s R_s:\rho'_1,\ldots,\rho_1^n,\nu_1,0,\ldots,0\right],\ldots,\left[\mu_r+k'_r R_1+\ldots+k_r^s R_s:\rho'_r,\ldots,\rho_r^n,0,\ldots,0,\nu_r\right],$$
$$\left[\mu_1-\beta_1+k'_1 R_1+\ldots+k_1^s R_s:\rho'_1,\ldots,\rho_1^n,\nu_1,0,\ldots,0\right],\ldots,\left[\mu_r-\beta_r+k'_r R_1+\ldots+k_r^s R_s:\rho'_r,\ldots,\rho_r^n,0,\ldots,0,\nu_r\right],$$

$$\left[\mu_1+\eta_1-\beta_1+k'_1 R_1+\ldots+k_1^s R_s:\rho'_1,\ldots,\rho_1^n,\nu_1,0,\ldots,0\right],\ldots,\left[\mu_r+\eta_r-\beta_r+k'_r R_1+\ldots+k_r^s R_s:\rho'_r,\ldots,\rho_r^n,0,\ldots,0,\nu_r\right]:$$
$$\left[\mu_1+\eta_1-\alpha_1+k'_1 R_1+\ldots+k_1^s R_s:\rho'_1,\ldots,\rho_1^n,\nu_1,0,\ldots,0\right],\ldots,\left[\mu_r+\eta_r-\alpha_r+k'_r R_1+\ldots+k_r^s R_s:\rho'_r,\ldots,\rho_r^n,0,\ldots,0,\nu_r\right]:$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];-;\ldots;-;\\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];-;\ldots;-;\end{matrix} Z_1,\ldots,Z_n,\frac{-x_1^{\nu_1 m_1}}{\xi_1},\ldots,\frac{-x_r^{\nu_r m_r}}{\xi_r}\right),$$

provided that $Z_i=\dfrac{z_i x_1^{\rho_1^i m_1}\ldots x_r^{\rho_r^i m_r}}{\xi_1^{\sigma_1^i}\ldots\xi_r^{\sigma_r^i}}$, $0\le\alpha_i<1, m_i\in N;\beta_i,\eta_i,x_i\in R;\mu_i>max(0,\beta_i-\eta_i)$,

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,\ldots,n\;;\;\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,$$

$i=n+1,\ldots,n+r$.

$F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([41],[42]) while $S_L^{h_1,\ldots,h_s}(x_1,\ldots,x_s)$ are generalized multivariable polynomials due to Srivastava and Garg [38], defined by (1.9).

(2.5) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\ldots x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}\right.$$

$$H_{A,C:\left[B',D'\right];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda:\left[\mu',\nu'\right];\ldots;\left[\mu^{(n)},\nu^{(n)}\right]}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.$$

$$\left.z_1 x_1^{m_1\rho_1'}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1'}\ldots x_r^{m_r\rho_r^1}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r'},\ldots,z_n x_1^{m_1\rho_1^n}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1^n}\ldots x_r^{m_r\rho_r^n}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)$$

$$\left.S_L^{h_1,\ldots,h_s}\left(w_1 x_1^{k'_1 m_1}\ldots x_r^{k'_r m_r},\ldots,w_s x_1^{k_1^s m_1}\ldots x_r^{k_r^s m_r}\right)\right\}$$

$$=\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}x_1^{(\mu_1-\beta_1-1)m_1}\ldots x_r^{(\mu_r-\beta_r-1)m_r}\sum\frac{(-1/\xi_1)^{N_1}}{N_1!}\cdots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\sum_{R_1,\ldots,R_s=0}^{h_1R_1+\ldots+h_sR_s\le L}(-L,h_1R_1+\ldots+h_sR_s)A(L;R_1,\ldots,R_s)\frac{w_1^{R_1}}{R_1!}\cdots\frac{w_s^{R_s}}{R_s!}x_1^{(\nu_1N_1+k_1'R_1+\ldots+k_1^sR_s)m_1}\ldots$$

$$x_r^{(\nu_rN_r+k'_rR_1+\ldots+k_r^sR_s)m_r}H^{0,\lambda+3r:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+3r,C+3r:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$\begin{matrix}[1+\lambda_1-N_1:\sigma'_1,\ldots,\sigma_1^n],[1-\mu_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\\ [1+\lambda_1:\sigma'_1,\ldots,\sigma_1^n],[1-\mu_1-\nu_1N_1+\beta_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\end{matrix}$$

$$\begin{matrix}[1-\mu_1-\eta_1+\beta_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\ldots,\\ [1-\mu_1-\eta_1+\alpha_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\ldots,\end{matrix}$$

$$\begin{matrix}[1+\lambda_r-N_r:\sigma'_r,\ldots,\sigma_r^n],[1-\mu_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n],\\ [1+\lambda_r:\sigma'_r,\ldots,\sigma_r^n],[1-\mu_r-\nu_rN_r+\beta_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n],\end{matrix}$$

$$\begin{matrix}[1-\mu_r-\eta_r+\beta_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n]:\\ [1-\mu_r-\eta_r+\alpha_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n]:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}Z_1,\ldots,Z_n\right),$$

where $Z_i=\dfrac{z_ix_1^{\rho_1^im_1}\ldots x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}\ldots\xi_r^{\sigma_r^i}}$, $min(\nu_1,\ldots,\nu_r;\rho_1^i,\ldots,\rho_r^i;\sigma_1^i,\ldots,\sigma_r^i)>0$, $0\le\alpha_j<1,m_j\in N;$ $\beta_j,\eta_j,x_j\in R;\mu_j>max(0,\beta_j-\eta_j),j=1,\ldots,r$ and $S_L^{h_1,\ldots,h_s}(w_1,\ldots,w_s)$ are generalized multivariable polynomials due to Srivastava and Garg [38] defined by (1.9). Our result (2.5) among others also includes (2.4) as special case.

3. Proofs of the Key Formulas. In this section, we prove the results of Section 2.

Proof of (2.1). For brevity, we denote

$$S\equiv\sum_{M_1,\ldots,M_n=0}^{\infty}\frac{1}{M!\ldots M_n!}\frac{\prod_{j=1}^{A}(a_j,M_1\theta'_j+\ldots+M_n\theta_j^{(n)})\prod_{j=1}^{B'}(b'_j,M_1\phi'_j)\ldots\prod_{j=1}^{B^{(n)}}(b_j^{(n)},M_n\phi_j^{(n)})}{\prod_{j=1}^{C}(c_j,M_1\psi'_j+\ldots+M_n\psi_j^{(n)})\prod_{j=1}^{D'}(d'_j,M_1\delta'_j)\ldots\prod_{j=1}^{D^{(n)}}(d_j^{(n)},M_n\delta_j^{(n)})}.$$

Therefore, **left hand side of (2.1)**

$$= S z_1^{M_1} \dots z_n^{M_n} D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \left\{ x_1^{(\mu_1 - 1 + \rho'_1 M_1 + \dots + \rho_1^n M_n) m_1} \left(x_1^{\nu_1 m_1} + \xi_1 \right)^{\lambda_1 - (\sigma_1 M_1 + \dots + \sigma_1^n M_n)} \dots \right.$$

$$\left. x_r^{(\mu_{r-1} + \rho'_r M_1 + \dots + \rho_r^n M_n) m_r} \left(x_r^{\nu_r m_r} + \xi_r \right)^{\lambda_r - (\sigma'_r M_1 + \dots + \sigma_r^n M_n)} \right\}$$

$$= S \frac{z_1^{M_1} \dots z_n^{M_n} \xi_1^{\lambda_1} \dots \xi_r^{\lambda_2}}{\xi_1^{\sigma'_1 M_1 + \dots + \sigma_1^n M_n} \dots \xi_r^{\sigma'_r M_1 + \dots + \sigma_r^n M_n}} \sum_{N_1, \dots, N_r = 0}^{\infty} \frac{(-1/\xi_1)^{N_1}}{N_1 !} \dots \frac{(-1/\xi_r)^{N_r}}{N_r !}$$

$$\left(\sigma'_1 M_1 + \dots + \sigma_1^n M_n - \lambda_1, N_1 \right) \dots \left(\sigma'_r M_1 + \dots + \sigma_r^n M_n - \lambda_r, N_r \right)$$

$$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \left\{ x_1^{(\mu_1 - 1 + \rho'_1 M_1 + \dots + \rho_1^n M_n + \nu_1 N_1) m_1} \dots x_r^{(\mu_r - 1 + \rho'_r M_1 + \dots + \rho_r^n M_r + \nu_r N_r) m_r} \right\}$$

$$= \prod_{j=1}^{r} \frac{\xi_j^{\lambda_j} \Gamma(\mu_j) \Gamma(\mu_j + \eta_j - \beta_j)}{\Gamma(\mu_j - \beta_j) \Gamma(\mu_j + \eta_j - \alpha_j)} x_j^{(\mu_j - \beta_j - 1) m_j} S \sum_{N_1, \dots, N_r = 0}^{\infty} \prod_{j=1}^{r} \frac{(\mu_j, \rho'_j M_1 + \dots + \rho_j^n M_n + \nu_j N_j)}{(\mu_j - \beta_j, \rho'_j M_1 + \dots + \rho_j^n M_n + \nu_j N_j)}$$

$$\frac{(\mu_j + \eta_j - \beta_j, \rho'_j M_1 + \dots + \rho_j^n M_n + \nu_j N_j)}{(\mu_j + \eta_j - \alpha_j, \rho'_j M_1 + \dots + \rho_j^n M_n + \nu_j N_j)} \frac{(-\lambda_j, \sigma_j^1 M_1 + \dots + \sigma_j^n M_n + 1)}{(-\lambda_j, \sigma'_j M_1 + \dots + \sigma_j^n M_n)}$$

$$\cdot \frac{\left(\dfrac{z_1 x_1^{\rho'_1 m_1} \dots x_r^{\rho'_r m_r}}{\xi_1^{\sigma'_1} \dots \xi_r^{\sigma'_r}} \right)^{M_1}}{M_1 !} \dots \frac{\left(\dfrac{z_n x_1^{\rho_1^n m_1} \dots x_r^{\rho_r^n m_r}}{\xi_1^{\sigma_1^n} \dots \xi_r^{\sigma_r^n}} \right)^{M_n}}{M_n !} \frac{\left(\dfrac{-x_1^{m_1}}{\xi_1} \right)^{N_1}}{N_1 !} \dots \frac{\left(\dfrac{-x_r^{m_r}}{\xi_r} \right)^{N_r}}{N_r !}$$

[By making an appeal to (1.5)],

which can be written in the form of right hand side of (2.1).

Proof of (2.2). For brevity, we denote

$$I \equiv \frac{1}{(2\pi\omega)^n} \int_{L_1} \dots \int_{L_n} \frac{\prod_{j=1}^{\lambda} \Gamma\left(1 - a_j + \sum_{i=1}^{n} \theta_j^{(i)} \zeta_i\right)}{\prod_{j=\lambda+1}^{A} \Gamma\left(a_j - \sum_{i=1}^{r} \theta_j^{(i)} \zeta_i\right) \prod_{j=1}^{C} \Gamma\left(1 - c_j + \sum_{i=1}^{r} \psi_j^{(i)} \zeta_i\right)}$$

$$\prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \zeta_i\right) \prod_{j=1}^{\nu^{(i)}} \Gamma\left(1 - b_j^{(i)} + \phi_j^{(i)} \zeta_i\right)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \zeta_i\right) \prod_{j=1}^{B^{(i)}} \Gamma\left(b_j^{(i)} - \phi_j^{(i)} \zeta_i\right)}.$$

Therefore, **left hand side of (2.2)**

$$= I\frac{z_1^{\zeta_1}\dots z_n^{\zeta_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\dots+\sigma''_1\zeta_n}\dots\xi_r^{\sigma'_r\zeta_1+\dots+\sigma''_r\zeta_n}}$$

$$\sum_{N_1,\dots,N_r=0}^{\infty}\frac{(-1)^{N_1+\dots+N_r}\left(\sigma'_1\zeta_1+\dots+\sigma''_1\zeta_n-\lambda_1,N_1\right)\dots\left(\sigma'_r\zeta_1+\dots+\sigma''_r\zeta_n-\lambda_r,N_r\right)}{\xi_1^{N_1}\dots\xi_r^{N_r}N_1!\dots N_r!}$$

$$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{\left(\mu_1-1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1\right)m_1}\dots x_r^{\left(\mu_r-1+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_nN_r\right)m_r}\right\}$$

$$= I\frac{z_1^{\zeta_1}\dots z_n^{\zeta_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\dots+\sigma''_1\zeta_n}\dots\xi_r^{\sigma'_r\zeta_1+\dots+\sigma''_r\zeta_n}}\sum_{N_1,\dots,N_r=0}^{\infty}\frac{(-1/\xi_1)^{N_1}}{N_1!}\dots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\frac{\Gamma\left(-\lambda_1+N_1+\sigma'_1\zeta_1+\dots+\sigma''_1\zeta_n\right)}{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\dots+\sigma''_1\zeta_n\right)}\dots\frac{\Gamma\left(-\lambda_r+N_r+\sigma'_r\zeta_1+\dots+\sigma''_r\zeta_n\right)}{\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\dots+\sigma''_r\zeta_n\right)}$$

$$\frac{\Gamma\left(\mu_1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1\right)\Gamma\left(\mu_1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1+\eta_1-\beta_1\right)}{\Gamma\left(\mu_1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1-\beta_1\right)\dots\Gamma\left(\mu_1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1+\eta_1-\alpha_1\right)}\dots$$

$$\frac{\Gamma\left(\mu_r+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_rN_r\right)\Gamma\left(\mu_r+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_rN_r+\eta_r-\beta_r\right)}{\Gamma\left(\mu_r+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_rN_r-\beta_r\right)\dots\Gamma\left(\mu_r+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_rN_r+\eta_r-\alpha_r\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1\zeta_1+\dots+\rho''_1\zeta_n+\nu_1N_1\right)m_1}\dots x_r^{\left(\mu_r-\beta_r-1+\rho'_r\zeta_1+\dots+\rho''_r\zeta_n+\nu_rN_r+\eta_r-\alpha_r\right)m_r}$$ (By an use of (1.5)],

which can be reformed in the form of right hand side of (2.2).

Proof. of (2.3). For brevity, if we denote

$$K\equiv S\frac{z_1^{M_1}\dots z_n^{M_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1M_1+\dots+\sigma''_1M_n}\dots\xi_r^{\sigma'_rM_1+\dots+\sigma''_rM_n}}\sum_{N_1,\dots,N_r=0}^{\infty}\frac{(-1/\xi_1)^{N_1}}{N_1!}\dots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\left(\sigma'_1M_1+\dots+\sigma''_1M_n-\lambda_1,N_1\right)\dots\left(\sigma'_rM_1+\dots+\sigma''_rM_n-\lambda_r,N_r\right)$$

$$\sum_{R_1,\dots,R_s=0}^{\infty}\frac{\prod_{j=1}^{E}\left(e_j,R_1\alpha'_j+\dots+R_s\alpha_j^{(s)}\right)\prod_{j=1}^{F'}\left(f'_j,R_1\beta'_j\right)\dots\prod_{j=1}^{F^{(s)}}\left(f_j^{(s)},R_s\beta_j^{(s)}\right)}{\prod_{j=1}^{G}\left(g_j,R_1\gamma'_j+\dots+R_s\gamma_j^{(s)}\right)\prod_{j=1}^{H'}\left(h'_j,R_1\eta'_j\right)\dots\prod_{j=1}^{H^{(s)}}\left(h_j^{(s)},R_s\psi_j^{(s)}\right)}\frac{w_1^{R_1}}{R_1!}\dots\frac{w_s^{R_s}}{R_s!}.$$

The left hand side of (2.3)

$$= KD_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{(\mu_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)m_1}\dots x_r^{(\mu_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)m_r}\right\}$$

$$= K\frac{(\mu_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)}{\Gamma(\mu_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)}$$

$$\frac{(\mu_1+\eta_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+\dots+k'_1 R_1+\dots+k_1^s R_s)}{(\mu_1+\eta_1-\alpha_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+\dots+k'_1 R_1+\dots+k_1^s R_s)}\dots$$

$$\frac{(\mu_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)}{(\mu_r-\beta_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)}$$

$$\frac{(\mu_r+\eta_r-\beta_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)}{(\mu_r+\eta_r-\alpha_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)}$$

$$x_1^{(\mu_1-\beta_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)m_1}\dots x_r^{(\mu_r-\beta_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)m_r}$$

[By making an appeal to (1.5)],

which can be adjusted in the form of right hand side of (2.3).

Proof of (2.4). For brevity considering

$$M \equiv S\frac{z_1^{M_1}\dots z_n^{M_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1 M_1+\dots+\sigma_1^n M_n}\dots\xi_r^{\sigma'_r M_1+\dots+\sigma_r^n M_n}}\sum_{N_1,\dots,N_r=0}^{\infty}(\sigma'_1 M_1+\dots+\sigma_1^n M_n-\lambda_1,N_1)\dots$$

$$(\sigma'_r M_1+\dots+\sigma_r^n M_n-\lambda_r,N_r)\frac{(-1/\xi_1)^{N_1}}{N_1!}\dots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\sum_{R_1,\dots,R_s=0}^{h_1R_1+\dots+h_sR_s\le L}(-L)_{h_1R_1+\dots+h_sR_s}A(L;R_1,\dots,R_s)\frac{w_1^{R_1}}{R_1!}\dots\frac{w_s^{R_s}}{R_s!}.$$

Left hand side of (2.4)

$$= MD_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{(\mu_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)m_1}\dots x_r^{(\mu_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s)m_r}\right\}$$

$$= M\frac{(\mu_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)}{\Gamma(\mu_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s)}$$

$$\frac{\left(\mu_1+\eta_1-\beta_1,\rho'_1 M_1+\ldots+\rho''_1 M_n+\nu_1 N_1+\ldots+k'_1 R_1+\ldots+k_1^s R_s\right)}{\left(\mu_1+\eta_1-\alpha_1,\rho'_1 M_1+\ldots+\rho''_1 M_n+\nu_1 N_1+\ldots+k'_1 R_1+\ldots+k_1^s R_s\right)}\ldots$$

$$\frac{\left(\mu_r,\rho'_r M_1+\ldots+\rho''_r M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\left(\mu_r-\beta_r,\rho'_r M_1+\ldots+\rho''_r M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$\frac{\left(\mu_r+\eta_r-\beta_r,\rho'_r M_1+\ldots+\rho''_r M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\left(\mu_r+\eta_r-\alpha_r,\rho'_r M_1+\ldots+\rho''_r M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1 M_1+\ldots+\rho''_1 M_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)m_1}\ldots x_r^{\left(\mu_r-\beta_r-1+\rho'_r M_1+\ldots+\rho''_r M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)m_r}$$

[By making an appeal to (1.5)],

which can be reformed as right hand side of (2.4).

Proof of (2.5). We can write

Left hand side of (2.5)

$$=I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma''_1\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma''_r\zeta_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{(-1/\xi_1)^{N_1}}{N_1!}\ldots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma''_1\zeta_n,N_1\right)\ldots\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma''_r\zeta_n,N_r\right)$$

$$\sum_{R_1,\ldots,R_s=0}^{h_1R_1+\ldots+h_sR_s\le L}(-L,h_1R_1+\ldots+h_sR_s)A(L;R_1,\ldots,R_s)\frac{w_1^{R_1}}{R_1!}\ldots\frac{w_s^{R_s}}{R_s!}$$

$$\left\{x_1^{\left(\mu_1-1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_1N_1+k'_1R_1+\ldots+k_1^sR_s\right)m_1}\ldots x_r^{\left(\mu_r-1+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_rN_r+k'_rR_1+\ldots+k_r^sR_s\right)m_r}\right\}$$

$$=I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma''_1\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma''_r\zeta_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\sum_{R_1,\ldots,R_0=0}^{h_1R_1+\ldots+h_sR_s\le L}(-L,h_1R_1+\ldots+h_sR_s)A(L;R_1,\ldots,R_s)$$

$$\frac{w_1^{R_1}}{R_1!}\ldots\frac{w_s^{R_s}}{R_s!}\frac{(-1/\xi_1)^{N_1}}{N_1!}\ldots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\frac{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma''_1\zeta_n+N_1\right)\ldots\Gamma\left(-\lambda_r\sigma'_r\zeta_1+\ldots+\sigma''_r\zeta_n+N_r\right)}{\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma''_1\zeta_n\right)\ldots\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma''_r\zeta_n\right)}$$

$$\frac{\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_1N_1+k'_1R_1+\ldots+k_1^sR_s\right)}{\Gamma\left(\mu_1-\beta_1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_1N_1+k'_1R_1+\ldots+k_1^sR_s\right)}$$

$$\frac{\Gamma\left(\mu_1+\eta_1-\beta_1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)}{\Gamma\left(\mu_1+\eta_1-\alpha_1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)}\cdots$$

$$\frac{\Gamma\left(\mu_r+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\Gamma\left(\mu_r-\beta_r+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$\frac{\Gamma\left(\mu_r+\eta_r-\beta_r+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\Gamma\left(\mu_r+\eta_r-\alpha_r+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1\zeta_1+\ldots+\rho''_1\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_1^s R_s\right)m_1}\ldots x_r^{\left(\mu_r-\beta_r-1+\rho'_r\zeta_1+\ldots+\rho''_r\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)m_r}$$

[By making an application of (1.5)],

which can be expressed in the form of right hand side of (2.5).

3. Special Cases. In this section, we mention the special cases of our results.

Special Cases of (2.1)

(a) For $m_i=1,\beta_i=\alpha_i, i=1,\ldots,r;$ (2.1) reduces to Chandel and Kumar [12,p.107, (2.1)].

(b) For $r=1, m_1=1, \beta_1=\alpha_1$, (2.1) gives Srivastava, Chandel and Vishwakarma [40, p. 567 (3.1)].

(c) For $r=2, m_1=m_2=1, \beta_1=\alpha_1, \beta_2=\alpha_2$, (2.1) reduces to Srivastava, Chandel and Vishwakarma [40, p. 568 (3.3)].

Special Cases of (2.2)

(a) For $m_i=1,\beta_i=\alpha_i; i=1,\ldots,r;$ (2.2) reduces to Chandel and Kumar [12, p.108 (2.3)], which also generalizes the earlier results due to Srivastava, Chandel and Vishwakarma [40, p. 563 (2.1), p. 564 (2.3)].

Special Cases of (2.3).

(a) For $m_i=1,\beta_i=\alpha_i, i=1,\ldots,r;$ (2.3) reduces to the result due to Chandel and Kumar [12, p.107, (2.2)].

(b) For $r=1, m_1=1, \beta_1=\alpha_1$, (2.3) reduces to the improved version of the result due to Srivastava, Chandel and Vishwakrama [40, p. 567 (3.2)]

(c) Due to general nature of the functions involved in (2.3), specializing the number of variables and other parameters in (2.3), we can get several interesting known and unknown results.

Special Cases of (2.4).

(a) For $r=1, \lambda_1=0, n=1, \sigma_1=0$, the result (2.4) will include the result due to Ram

and Chandak [32, p.53 (10)] involving Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ and generalized multi-variable polynomials $S_L^{h_1,\ldots,h_s}$ of Srivastava and Garg [38].

(b) For $r=1, \lambda_1=0, n=1, \sigma'_1=0$ and $\beta_1=\alpha_1$, (2.4) includes an interesting result due to Ram-Chandak [32, p. 53 (11)] for the Riemann-Liouville derivative operator-defined by Miller and Ross [28].

(c) For $s=1$, the polynomials $S_L^{h_1,\ldots,h_s}(x_1,\ldots,x_s)$ reduce to the Srivastava polynomials S_l^h defined by (1.10), therefore, for $r=1, \lambda_1=0, n=1, \sigma'_1=0, \beta_1=\alpha_1$ and $s_1=1$, (2.4) gives an interesing result due to Ram-Chandak [32, p. 54(12)] involving Srivastava polynomials.

(d) Similarly all the following results due to Ram and Chandak [32, p.54 (13), (14), (15)] are included in our result (2.4). Since for $l=0, A_{0,0}=1, S_l^h(x)=1$, therefore our result (2.4), also gives the result due to Kilbas [21] as special case of Ram and Chandak [32 (15)] .

(e) Since Wright function $\phi(\alpha,\beta;z)$ defined by (1.7) and Wright generalized Bessel function $J_\nu^\delta(z)$ defined by (1.8) are special cases of Fox-Wright generalized hypergeometric function ${}_p\Psi_q(z)$ defined by (1.6), therefore, all the results due to Ram and Chandak [32, pp. 55-57 (16), (17), (18),(19)(20),(21)] are included in our result (2.4) as special cases.

Special Cases of (2.5). Our result (2.5) includes (2.4) along with all results of Ram and Chandak [32] as special cases.

4. Other Special Cases as Application of one Fractional Derivative. In this section, making an appeal to (1.5) and one fractional derivative we derive

$$(4.1)\ D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} S_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\, y_1x^{\lambda_1 m},\ldots,y_nx^{\lambda_n m}\right)\right\}$$

$$= x^{(\mu-1-\beta)m} S_{C+2:D';\ldots;D^{(n)}}^{A+2:\beta';\ldots;\beta^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[\mu:\lambda_1,\ldots,\lambda_n],[\mu+\eta-\beta:\lambda_1,\ldots,\lambda_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-\beta:\lambda_1,\ldots,\lambda_n],[\mu+\eta-\alpha:\lambda_1,\ldots,\lambda_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];...;\left[\left(b^{(n)}\right):\phi^{(n)}\right] \\ [(d'):\delta'];...;\left[\left(d^{(n)}\right):\delta^{(n)}\right]\end{matrix} y_1x^{\lambda_1 m},...,y_nx^{\lambda_n m}\right),$$

provided that $0 \le \alpha < 1, m \in N; \beta, \eta, \lambda_i > 0, x \in R, \mu > max(1, \beta - \eta)$ and

$$1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0 \;\; i = 1,...,n.$$

(4.2) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} F_D^{(n)}\left(\mu-\beta, b_1,...,b_n; \mu; y_1x^m,...,y_nx^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} F_D^{(n)}\left(\mu+\eta-\beta, b_1,...,b_n; \mu+\eta-\alpha; y_1x^m,...,y_nx^m\right)$$

valid if $0 \le \alpha < 1, m \in N; \beta, \eta, x \in R, k > max(0, \beta - \eta - 1)$, $|y_1x| < 1,...,|y_nx| < 1$,

where $F_D^{(n)}$ is well known Lauricella function [24].

(4.3) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} F_D^{(n)}\left(\mu+\eta-\alpha, b_1,...,b_n, \mu+\eta-\beta; y_1x^m,...,y_nx^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} F_D^{(n)}\left(\mu, b_1,...,b_n; \mu-\beta; y_1x^m,...,y_nx^m\right),$$

provided that all conditions of (4.2) are satisfied.

(4.4) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} F_D^{(n)}\left(\mu+\eta-\alpha, b_1,...,b_n, \mu; y_1x^m,...,y_nx^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} F_D^{(n)}\left(\mu+\eta-\beta, b_1,...,b_n; \mu-\beta; y_1x^m,...,y_nx^m\right),$$

where all conditions of (4.2) hold true.

(4.5) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} F_D^{(n)}\left(\mu-\beta, b_1,...,b_n; \mu+\eta-\beta; y_1x^m,...,y_nx^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} F_D^{(n)}\left(\mu, b_1,...,b_n; \mu+\eta-\beta; y_1x^m,...,y_nx^m\right),$$

valid if all conditions of (4.2) are satisfied.

(4.6) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, exp(x^m)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_2F_2\left(\mu,\mu+\eta-\beta;\mu-\beta,\mu+\eta-\alpha;x^m\right),$$

provided that $0 \le \alpha < 1, m \in N, \beta, \eta, x \in R, \mu > max(1, \beta-\eta)$.

(4.7) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_2\left(\mu-\beta,\mu+\eta-\alpha;\mu,\mu+\eta-\beta;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} exp\left(x^m\right),$$

where all conditions of (4.6) are satisfied.

(4.8) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_1\left(\mu-\beta;\mu;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_1F_1\left(\mu+\eta-\beta;\mu+\eta-\alpha;x^m\right),$$

which holds true if all conditions of (4.6) are satisfied.

(4.9) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_1\left(\mu+\eta-\alpha;\mu;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_1F_1\left(\mu+\eta-\beta;\mu-\beta;x^m\right),$$

where all conditions of (4.6) hold true.

(4.10) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_1\left(\mu-\beta;\mu+\eta-\alpha;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_1F_1\left(\mu;\mu+\eta-\alpha;x^m\right),$$

provided that all conditions of (4.6) hold time.

(4.11) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_1\left(\mu+\eta-\alpha;\mu+\eta-\beta;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_1F_1\left(\mu;\mu-\beta;x^m\right),$$

which is true if all conditions of (4.6) are setisfied.

(4.12) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_1\left(\mu-\beta;\mu+\eta-\alpha;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)}$$

valid if all conditions of (4.6) are satisfied.

(4.13) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_1\left(\mu-\beta;\mu+\eta-\alpha,\mu+\eta-\beta;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-\mu},$$

where all conditions of (4.6) are satisfied.

(4.14) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_1\left(\mu-\beta;\mu-\eta-\alpha,\mu;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)},$$

provided that all conditions of (4.6) hold true.

(4.15) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_{2.}\left(\mu-\beta;\mu,\mu+\eta-\beta;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)},$$

valid if all conditions of (4.6) are true.

(4.16) $D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_2\left(\mu+\eta-\alpha;\mu,\mu+\eta-\beta;x^m\right)\right\}$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_0F_1\left(-;\mu-\beta;x^m\right),$$

which holds true if all conditions of (4.6) are satisfied.

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STRONG CONVERGENCE THEOREMS FOR UNIFORMLY EQUI-CONTINUOUS AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

By

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ABSTRACT

The purpose of this paper is prove some strong convergence theorems of the modified Ishikawa iterative sequences with errors for unformly equi-continuous and asymptotically quasi-nonexpansive mapping in the setup of unifromly convex Banach spaces by using condition (A) instead of comletely continuous or demicompact condition. Our results improve and generalize the corresponding results of Rhoades [6], Schu [7,8]. Tan and Xu [10,11] Xu and Noor [12] and many others.

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1. Introduction and Preliminaries. Let E be a real normed linear space, K be a nonempty subset of E. Throughout the paper, N denotes the set of positive integers and $F(T)=\{x:Tx=x\}$ the set of fixed points of a mapping T. Let $T:K\to K$ be a given mapping.

(1) T is said to be asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$\|T^n x-T^n y\|\leq k_n\|x-y\|, \tag{1.1}$$

for all $x,y\in K$ and $n\in N$.

(2) T is said to be asymptotically quasi-nonexpansive if $F(T)\neq\phi$ and there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_n k_n=1$ such that

$$\|T^n x - p\| \le k_n \|x - p\|; \tag{1.2}$$

for all $x \in K, p \in F(T)$ and $n \in N$.

(3) T is said to be uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \tag{1.3}$$

for all $x, y \in K$ and $n \in N$.

(4) T is said to be uniformly Holder continuous [5] if there exist positive constants L and α such that

$$\|T^n x - T^n y\| \le L\|x - y\|^\alpha, \tag{1.4}$$

for all $x, y \in K$ and $n \in N$.

(5) T is said to be uniformly equi-continuous [5] if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|T^n x - T^n y\| \le \epsilon \tag{1.5}$$

whenever $\|x - y\| < \delta$ for all $x, y \in K$ and $n \ge 1$ or, equivalently, T is uniformly equi-continuous if and only if $\|T^n x_n - T^n y_n\| \to 0$ whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Remark 1.1 (i) It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian.
(ii) If T is uniformly L-Lipschitzian, then it is uniformly Holder continuous with constants L > 0 and $\alpha = 1$.
(iii) If T is uniformly Holder continuous, then it is uniformly equi-continuous.

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that, if K is a nonempty bounded asymptotically nonexpansive self-mapping of K has a fixed point. Moreover the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied weak and strong convergence problem of the Mann and Ishikawa iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [2,6,7,8,10,11,12] and references therein).

Recenyly, Liu [3] studied modified Ishikawa iterative sequences with errors for asymptotically quasi-nonexpansive and unifornly Holder continuous mappings in uniformly convex Banach spaces and established some strong convergence theorems which extended some corresponding results of Tan and Xu [11].

The purpose of this paper is to extend and improve some resuits of [3] for

uniformly equi-continuous and asymptotically quasi-nonexpansive mappings. Also our results improve and generalize the corresponding results of [6,7,8,11,12,] and many others.

In order to prcve the main results in this paper, we need the following lemmas :

Lemma 1.1. (Tan and Xu [10]). *Let* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegtive numbers satisfying the inequality

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \forall_n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.2. ([1]) Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X, Then there exists a continuous increasing convex function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|)$$

for all $x, y \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda+\mu+\gamma=1$.

2. Main Results. Now, we give the main results of this paper.

Lemma 2.1. Let E be a normed linear space and K be a nonempty convex subset of E. Let $T{:}K\to K$ be an asymptotically quasi-nonexpansive mapping with $F(T)\neq\phi$. and a seqeunce $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$. Let $\{x_n\}$ be a sequence in K defined by

$$y_n = a_n' x_n + b_n' T^n x_n + c_n' v_n,$$

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad n \geq 1, \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in E and $\{a_n\},\{b_n\},\{c_n\},\{a_n'\},\{b_n'\},\{c_n'\}$ are sequences in $[0,1]$ and $a_n+b_n+c_n= a_n'+b_n'+c_n'=1$ with the restrictions $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n c_n' < \infty$. Then we have the following:

(i) $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$,

(ii) $\lim_{n\to\infty} d(x_n, F(T))$ exists, where $d(x,F(T))$ denotes the distance from x to the set $F(T)$.

Proof of (i). Let $p \in F(T)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. So we can set

$$M = \max\left\{\sup_{n\ge1}\|u_n - p\|, \sup_{n\ge1}\|v_n - p\|\right\}.$$

Then it follows from (2.1) that

$$\begin{aligned}
\|y_n - p\| &= \|a'_n x_n + b'_n T^n x_n + c'_n v_n - p\| \\
&= \left\|a'_n (x_n - p) + b'_n (T^n x_n - p) + c'_n (v_n - p)\right\| \\
&\le a'_n \|x_n - p\| + b'_n \|T^n x_n - p\| + c'_n \|v_n - p\| \\
&\le a'_n \|x_n - p\| + b'_n k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&\le \left[a'_{,} + b'_n\right]_n k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&= \left[1 - c'_n\right] k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&\le k_n \|x_n - p\| + c'_n M. \qquad (2.2)
\end{aligned}$$

Again from (2.1) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\| \\
&= \left\|a_n (x_n - p) + b_n (T^n y_n - p) + c_n (u_n - p)\right\| \\
&\le a_n \|x_n - p\| + b_n \|T^n y_n - p\| + c_n \|u_n - p\| \\
&\le a_n \|x_n - p\| + b_n k_n \|y_n - p\| + c_n \|u_n - p\| \\
&\le a_n \|x_n - p\| + b_n k_n \left[k_n \|x_n - p\| + c'_n M\right] + c_n \|u_n - p\| \\
&\le \left[a_n + b_n\right] k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&= \left[1 - c_n\right] k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&\le k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&\le \left[1 + (k_n^2 - 1)\right] \|x_n - p\| + (b_n c'_n k_n + c_n) M \qquad (2.3)
\end{aligned}$$

since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n c'_n < \infty$, it follows from Lemma 1.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof of part (i).

Proof of (ii) From conclusion of part (i), we have

$$\|x_{n+1}-p\| \le \left[1+\left(k_n^2-1\right)\right]\|x_n-p\|+\left(b_n c_n' k_n+c_n\right)M.$$

This gives that

$$d\left(x_{n+1},F(T)\right) \le \left[1+\left(k_n^2-1\right)\right]d\left(x_n,F(T)\right)+\left(b_n c_n' k_n+c_n\right)M.$$

Since $\sum_{n=1}^{\infty}\left(k_n^2-1\right)<\infty, \sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}b_n c_n'<\infty$, it follows from Lemma 1.1, we know that $\lim_{n\to\infty} d\left(x_n,F(T)\right)$ exists. This completes the proof of part (ii).

Theorem 2.1. Let E be a uniformly convex Banach space, K be a nonempty convex subset of E and $T{:}K\to K$ be a uniformly equi-continuous and asymptotically quasi-nonexpansive mapping with $F(T)\neq\phi$ and a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ and $\sum_{n=1}^{\infty}\left(k_n^2-1\right)<\infty$. Let $\{x_n\}$be a sequence in K defined by (2.1) with the following restrictions:

(i) $0\le b_n\le b<1$ and $b_{n+1}\le b_n$ for all $n\ge 1$.

(ii) $\sum_{n=1}^{\infty}b_n=\infty$, (iii) $\lim_{n\to\infty}b_n'=0$, (iv) $\sum_{n=1}^{\infty}c_n=\infty$ and $\sum_{n=1}^{\infty}b_n c_n'<\infty$.

Then $\liminf_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. Since $T{:}K\to K$ is asymptotically quasi-nonexpansive, we have

$$\|T^n y_n-p\|\le k_n\|y_n-p\|\le k_n^2\|x_n-p\|$$

for any $p\in F(T)$. By Lemma 2.1(i), we know that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence $\{x_n-p\}$ and $\{T^n y_n-p\}$ are bounded sequences in E. Set $r_1=\sup\{\|x_n-p\|:n\ge1\}$, $r_2=\sup\{\|T^n y_n-p\|:n\ge1\}$, $r_3=\sup\{\|u_n-p\|:n\ge1\}$, $r_4=\sup\{\|v_n-p\|:n\ge1\}$ and $r=\max\{r_i:i=1,2,3,4\}$ for any fixed $p\in F(T)$. Then we have $\{x_n-p\},\{T^n y_n-p\}$, $\{u_n-p\},\{v_n-p\}\in B_r(0)$ for all $n\ge1$. By using Lemma 1.2 and (2.1), we have

$$\|y_n-p\|^2 = \left\|a_n'(x_n-p)+b_n'(T^n x_n-p)+c_n'(v_n-p)\right\|^2$$

$$\le a_n'\|x_n-p\|^2+b_n'\|T^n x_n-p\|^2+c_n'\|v_n-p\|^2-a_n'b_n'g\left(\|x_n-T^n x_n\|\right)$$

$$\le a_n'\|x_n-p\|^2+b_n'k_n^2\|x_n-p\|^2+c_n'r^2$$

$$\le \left[a_n' + b_n'\right] k_n^2 \left\|x_n - p\right\|^2 + c_n' r^2$$

$$= \left[1 - c_n'\right] k_n^2 \left\|x_n - p\right\|^2 + c_n' r^2$$

$$\le k_n^2 \left\|x_n - p\right\|^2 + c_n' r^2. \tag{2.4}$$

Again, using Lemma 1.2, (2.1) and (2.4) we obtain

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &= \left\|a_n (x_n - p) + b_n (T^n x_n - p) + c_n (u_n - p)\right\|^2 \\
&\le a_n \left\|x_n - p\right\|^2 + b_n \left\|T^n y_n - p\right\|^2 + c_n \left\|u_n - p\right\|^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&\le a_n \left\|x_n - p\right\|^2 + b_n k_n^2 \left\|y_n - p\right\|^2 + c_n \left\|u_n - p\right\|^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&\le a_n \left\|x_n - p\right\|^2 + b_n k_n^2 \left[k_n^2 \left\|x_n - p\right\|^2 + c_n' r^2\right] + c_n r^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&\le \left[a_n + b_n\right] k_n^4 \left\|x_n - p\right\|^2 + b_n c_n' k_n^2 r^2 + c_n r^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&= \left[1 - c_n\right] k_n^4 \left\|x_n - p\right\|^2 + b_n c_n' k_n^2 r^2 + c_n r^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&\le k_n^4 \left\|x_n - p\right\|^2 + \left(b_n c_n' k_n^2 + c_n\right) r^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right) \\
&\le \left[1 + \left(k_n^4 - 1\right)\right] \left\|x_n - p\right\|^2 + \left(b_n c_n' k_n^2 + c_n\right) r^2 - a_n b_n g\left(\left\|x_n - T^n y_n\right\|\right). \qquad (2.5)
\end{aligned}$$

Note that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ is equivalent $\sum_{n=1}^{\infty}(k_n^4 - 1) < \infty$ and so, setting $\rho_n = r^2\left(k_n^4 - 1\right)$, then $\sum_{n=1}^{\infty}\rho_n < \infty$. Furthermore, since $g:[0,\infty) \to [0,\infty)$ is a continuous increasing convex function and $\left\{x_n - T^n y_n\right\}$ is a bounded sequence in E, we assert that $g\left(\left\|x_n - T^n y_n\right\|\right)$ is bounded. Set $\sigma_n = c_n g\left(\left\|x_n - T^n y_n\right\|\right)$, we have $\sum_{n=1}^{\infty}\sigma_n < \infty$. Since $\{k_n\}$ is bounded, and by hypothesis $\sum_{n=1}^{\infty} b_n c_n' < \infty$ so $\sum_{n=1}^{\infty} b_n c_n' k_n^2 < \infty$. Now, set

$$\delta_n = \rho_n + \sigma_n + \left(b_n c_n' k_n^2 + c_n\right) r^2.$$

Then $\sum_{n=1}^{\infty}\delta_n < \infty$. By the assumption (i), we have $(1-b_n) \ge (1-b)$. It follows from (2.5) that

$$\left\|x_{n+1} - p\right\|^2 \le \left\|x_n - p\right\|^2 - (1 - b_n - c_n) b_n g\left(\left\|x_n - T^n y_n\right\|\right) + \rho_n \left(b_n c_n' k_n^2 + c_n\right) r^2$$

$$\leq \|x_n - p\|^2 - (1-b_n)b_n g\left(\|x_n - T^n y_n\|\right) + \rho_n + \sigma_n + \left(b_n c_n' k_n^2 + c_n\right) r^2$$

$$\leq \|x_n - p\|^2 - (1-b_n)b_n g\left(\|x_n - T^n y_n\|\right) + \delta_n \tag{2.6}$$

which leads to

$$(1-b)b_n g\left(\|x_n - T^n y_n\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n, \tag{2.7}$$

and

$$(1-b)b_{n+1} g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) \leq \|x_{n+1} - p\|^2 - \|x_{n+2} - p\|^2 + \delta_{n+1}, \text{for all } n \geq 1. \tag{2.8}$$

Adding on both sides of (2.7) and (2.8) and using the condition $b_{n+1} \leq b_n$ for all $n \geq 1$, we have

$$(1-b)\sum_{n=1}^{\infty} b_{n+1}\left[g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) + g\left(\|x_n - T^n y_n\|\right)\right] < \infty. \tag{2.9}$$

Since $\sum_{n=1}^{\infty} b_n = \infty$ by the assuinption (ii), we have

$$\liminf_{n\to\infty}\left[g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) + g\left(\|x_n - T^n y_n\|\right)\right] = 0. \tag{2.10}$$

By virtue of the continity and monotonicity of function g, we assert that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j} - T^{n_j} y_{n_j}\| \to 0, \|x_{n_{j+1}} - T^{n_{j+1}} y_{n_{j+1}}\| \to 0 \text{ as } j \to \infty. \tag{2.11}$$

By the assumption (iii), we see that

$$\|y_n - x_n\| \leq b_n' \|x_n - T^n x_n\| + c_n' \|v_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{2.12}$$

It follows from the uniform equi-continuity of T that

$$\|T^n y_n - T^n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Now we observe that

$$\|x_n - T^n x_n\| \leq \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\|. \tag{2.14}$$

It follows from (2.11) and (2.13) that

$$\|x_{n_j} - T^{n_j} x_{n_j}\| \to 0 \text{ and } \|x_{n_{j+1}} - T^{n_{j+1}} x_{n_{j+1}}\| \to 0, \text{ as } j \to \infty.$$

Since $\|T^{n_j} x_{n_j} - x_{n_j}\| \to 0$ as $j \to \infty$, we have

$$\|x_{n_{j+1}} - x_{n_j}\| \leq b_{n_j} \|T^{n_j} y_{n_j} - x_{n_j}\| + c_{n_j} \|u_{n_j} - x_{n_j}\| \to 0 \text{ as } j \to \infty. \tag{2.15}$$

It follows from the uniform equi-continuity of T that

$$\left\|T^{n_j} x_{n_{j+1}} - x_{n_j}\right\| \to 0 \text{ as } j \to \infty. \tag{2.16}$$

Again, from above inequalities, we observe that

$$\begin{aligned}\left\|T^{n_j} x_{n_{j+1}} - x_{n_{j+1}}\right\| &\le \left\|T^{n_j} x_{n_{j+1}} - T^{n_j} x_{n_j}\right\| + \left\|T^{n_j} x_{n_j} - x_{n_{j+1}}\right\| \\ &\le \left\|T^{n_j} x_{n_{j+1}} - T^{n_j} x_{n_j}\right\| + \left\|T^{n_j} x_{n_j} - x_{n_j}\right\| + \left\|x_{n_j} - x_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty. \end{aligned} \tag{2.17}$$

It follows from the uniform equi-continuity of T that

$$\left\|T^{n_j+1} x_{n_{j+1}} - T x_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty, \tag{2.18}$$

and

$$\left\|x_{n_{j+1}} - T x_{n_{j+1}}\right\| \le \left\|x_{n_{j+1}} - T^{n_{j+1}} x_{n_{j+1}}\right\| + \left\|T^{n_{j+1}} x_{n_{j+1}} - T x_{n_{j+1}}\right\|. \tag{2.19}$$

Therefore, it follows from (2.17), (2.18) and the above inequality that

$$\left\|x_{n_{j+1}} - T x_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty. \tag{2.20}$$

This completes the proof.

Let $\{z_n\}$ be a given sequence in K. Recall that a mapping $T{:}K \to K$ with $F(T) \neq \phi$ is said to satisfy *condition* (A) [9] if there exists a nondecreasing function with $f:[0,\infty) \to [0,\infty)$ with $f(0)=0$, $f(r)>0$ for all $r \in (0,\infty)$ such that

$$\|z_n - Tz_n\| \ge f\left(d\left(z_n, F(T)\right)\right) \text{ for all } n \ge 1,$$

where $d\left(z_n, F(T)\right) = \inf\left\{\|z_n - p\| : p \in F(T)\right\}$.

By using Theorem 2.1, we have the following :

Theorem 2.2. Let E be a uniformly convex Banach space, K be a nonempty convex subset of E and $T{:}K \to K$ be a unifromly equi-continuous and asymptotically quasi-nonexpansive mapping with $F(T) \neq \phi$ and a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1) with the following restrictions:

(i) $0 \le b_n \le b < 1$ and $b_{n+1} \le b_n$ for all $n \ge 1$,

(ii) $\sum_{n=1}^{\infty} b_n = \infty$, (iii) $\lim_{n\to\infty} b'_n = 0$, (iv) $\sum_{n=1}^{\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} b_n c'_n < \infty$.

If T satisfies *condition* (A), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from Theorem 2.1 that

$$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Since T satisfies condition (A), we have

$$\liminf_{n\to\infty} f\left(d\left(x_n, FT\right)\right) = 0.$$

From the property of f, it follows that

$$\liminf_{n\to\infty} d\left(x_n, F(T)\right) = 0.$$

It follows from Lemma 2.1 that $d(x_n, F(T)) \to 0$ as $n \to \infty$. Now, we can take an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T)$ such that $\|x_{n_j} - p_j\| \le 2^{-j}$. Set $M = \exp\left\{\sum_{n=1}^{\infty}\left(k_n^2 - 1\right)\right\}$ and write $n_j + 1 = n_{j+l}$ for some $l \ge 1$. Then we have

$$\begin{aligned}
\left\|x_{n_{j+1}} - p_j\right\| &= \left\|x_{n_j+l} - p_j\right\| \\
&\le k^2_{n_j+l-1}\left\|x_{n_j+l-1} - p_j\right\| \\
&\le \left[1 + \left(k^2_{n_j+l-1} - 1\right)\right]\left\|x_{n_j+l-1} - p_j\right\| \\
&\le \exp\left\{\left(k^2_{n_j+l-1} - 1\right)\right\}\left\|x_{n_j+l-1} - p_j\right\| \\
&\le \exp\left\{\sum_{m=0}^{l-1}\left(k^2_{n_j+m} - 1\right)\right\}\left\|x_{n_j} - p_j\right\| \\
&\le \frac{M}{2^j}. \qquad (2.21)
\end{aligned}$$

It follows from (2.21) that

$$\begin{aligned}
\|p_{j+1} - p_j\| &\le \left\|p_{j+1} - x_{n_j+1}\right\| + \left\|x_{n_j+1} - p_j\right\| \\
&\le \frac{1}{2^{j+1}} + \frac{M}{2^j} \\
&\le \frac{2M+1}{2^{j+1}}. \qquad (2.22)
\end{aligned}$$

Hence $\{p_j\}$ is a Cauchy sequence. Assume that $p_j \to p$ as $j \to \infty$. Then $p \in F(T)$ since $F(T)$ is closed, which implies that $x_j \to p$ as $j \to \infty$. This completes the proof.

Remark 2.1. We note that, if $T:K\to K$ is completely continuous, then it must be

demicompact [8], and if T is continuous and demicompact, it must satisfy *condition* (A) [4,9]. In view of this observation, Theorem 2.2 improves the corresponding result of Liu [3] in the following aspects:

(i) K may be not necessarily compact or bounded,

(ii) T may be not uniformly Holder continous.

Remark 2.2. Our results improve and generalize the corresponding results of [6,7,8,10,11,12] and many others from the existing literature.

Remark 2.3. Our results also extend the corresponding results of Cho et al. [1] to the case $\phi_0$continuous asymptotically quasi-nonexpansive mappings.

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IMPACT OF SEXUAL MATURATION ON THE TRANSMISSION DYNAMICS OF *HIV* INFECTION IN HETEROGENEOUS COMMUNITY : A MODEL AND ITS QUALITATIVE ANALYSIS

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ABSTRACT

In this paper, we develop a model to study the spread of *HIV* infection, which can cause Acquired Immunodificiency Syndrome (*AIDS)*, through Vertical and horizontal transmissions and introduce the concept of sexually immature and mature individuals by considering maturation rates in heterosexual community and partially analyzed. We obtain equilibrium points of the system at two states (Disease-free and Endemic). We investigate the criteria for existence of endemic steady state of the system. We determine local and global dynamics of these steady states of the system and conclude.

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Keywords: Vertical and Horizontal Transmissions, Heterosexual Community, Asymptotic Stability, Epidemiological Parameters.

1. Introduction. Today's, Acquired Immunodeficiency Syndrome (*AIDS*), has shown a very high degree of prevalence in populations all over the world, which is caused by Human Immunodeficiency Virus (*HIV*). The nature of human interactions, the uncertainties in the current estimates of epidemiological parameters and the lack of enough reliable data make it extremely difficult to understand the dynamics of the virus transmission without the frame works provided by mathematical modelling. The study of mathematical modelling is also helpful in determining the demographic and economic impact of the epidemic which

is turn help us to develop reasonable scientifically and socially sound investigation plans in order to reduce the spread of the infection.

In recent decades, several mathematical modelling studies have been conducted to describe the transmission dynamics of *HIV* infection for homogeneous and heterogeneous populations, Anderson et al. ([1][2]), Bailey [5], Knox [16], Pickering et al. [21], May and Anderson ([17][20]), Grant et al. [12], Hethcote [13], Anderson et al. [3], May [18] Castillo-Chavez, Cooke, Huang and Levin ([7],[8],[9]), Sun [23], Chen [10] and Hethcote [13].

In particular, Anderson et al. [1] described some preliminary attempts to use mathematical models for transmission of *HIV* in a homosexual community. May and Anderson [17] presented simple *HIV* transmission models to help clarify the effects of various factors on the overall pattern of *AIDS* epidemic. Blythe and Anderson (1988) considered *HIV* transmission models with four forms for the distribution of incubation period by assuming that the infectious period is equal to the incubation period. Castillo-Chavez et al. [7] analyzed a model where the mean rate of acquisition of new partners depends on the size of the sexually active population. Most of the above mentioned models consider only one population but *HIV* transmission takes place in the population that are heterogeneous in a variety of ways and this aspect should be taken in modelling *HIV*. Knox [16], Colgete et al. [11], Jacquez et al. [14], Koopman et al. [15].

2. Model Formulation. Let us consider a heterosexual community of size P with uniform promiscuous behaviour and taking only heterosexual encounters and assume that the birth and death rates are same, making community size to be a constant. We have assumed that infection passes in the considering population through the member of one male or female class to the other female or male class respectivley. The infection can also be transmitted vertically to the offspring of infected mother.

Let any instant of time t this considering community be subdivided into six classes of $S_1(t)$ mature male susceptibles, $I_1(t)$ mature male infective having *HIV* infection, $S_2(t)$ mature female susceptibles, $I_2(t)$ mature female infective having *HIV* infection, $X(t)$ immature susceptibles and $Y(t)$ immature infectives having *HIV* infection. The susceptibles become infected with transmssion efficiency k and immature susceptibles and infectives being sexually matured at rates m and m' respectively.

This leads to the following system of ordinary differential equations

$$\frac{dS_1}{dt} = -kS_1I_2 + \alpha mX - bS_1, \quad \frac{DI_1}{dt} = kS_1I_2 + \alpha m'Y - b'I_1,$$

$$\frac{DS_2}{dt} = -kS_2I_1 + (1-\alpha)mX - bS_2,\ \frac{dI_2}{dt} = kS_2I_1 + (1-\alpha)m'Y - b'I_2,$$

$$\frac{dX}{dt} = bS + pb'I - \alpha mX - (1-\alpha)mX,\ \frac{dY}{dt} = qb'I - \alpha m'Y - (1-\alpha)m'Y, \quad (2.1)$$

with initial data

$$S_1(0) = S_{10} > 0, S_2(0) = S_{20} > 0, I_1(0) = I_{10} > 0, I_2(0) = I_{20} > 0,\ X(0) = X_0 > 0,$$

$$Y(0) = Y_0 > 0, P = S + I + X + Y, S = S_1 + S_2, I = I_1 + I_2, 1 = p + q, 0 < \alpha < 1 \quad (2.2)$$

where

α and $(1-\alpha)$= The proportions of male and female hosts, who converts from immature class to mature class respectively.

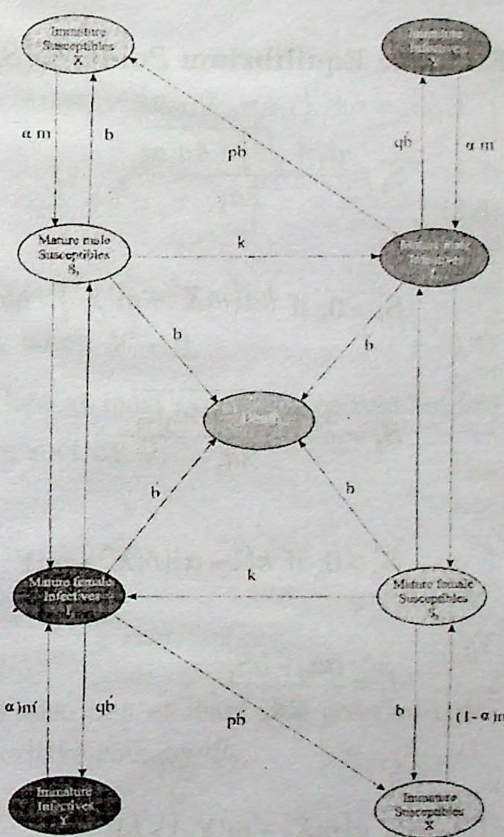
b and b' are Birth rtes of immature susceptible and infectives respectively.

P= The fraction of newborn offspring of infective parents, who are susceptible at birth.

q= The fraction of newborn offsprings of infective parents, who are infective at birth.

b and b' are also Death rate of mature susceptibles and infectives respectively.

The above mathematical modelling can be understood by Fig.1.



3. Equilibrium Points of Model. The equilibrium points of the model can be derived from the following set of equations

$$-kS_1^*I_2^* + \alpha mX^* - bS_1^* = 0,$$

$$kS_1^*I_2^* + \alpha m'Y^* - b'I_1^* = 0,$$

$$-kS_2^*I_1^* + (1-\alpha)mX^* - bS_2^* = 0,$$

$$kS_2^*I_1^* + (1-\alpha)m'Y^* - b'I_2^* = 0,$$

$$b\left(S_1^* + S_2^*\right) + pb'\left(I_1^* + I_2^*\right) - \alpha m X^* - (1-\alpha) m X^* = 0,$$

$$qb'\left(I_1^* + I_2^*\right) - \alpha m' Y^* - (1-\alpha) m' Y^* = 0,$$

$$S_1^* + S_2^* + I_1^* + I_2^* + X^* + Y^* = p. \tag{3.1}$$

Disease Free Equilibrium Point $E_0\left(S_1^*, I_1^*, S_2^*, I_2^*, X^*, Y^*\right)$.

$$I_1^* = I_2^* = Y^* = 0,$$

$$S_1^* = \frac{\alpha m P}{m+b}, S_2^* = \frac{(1-\alpha) m P}{m+b}, X = \frac{bP}{m+b},$$

Endemic Equilibrium Point $E_1\left(S_1^*, I_1^*, S_2^*, I_2^*, X^*, Y^*\right)$.

$$S_1^* = \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_1},$$

$$S_1^* > 0; \text{ if } ka\left(mX^* + m'Y^*\right) > bb' + \frac{km'}{q} Y^*,$$

$$S_2^* = \frac{a_4 + \sqrt{a_4^2 + 4a_1 a_5}}{2a_1}$$

$$S_2^* > 0; \text{ if } k(1-\alpha)\left(mX^* + m'Y^*\right) > bb' + \frac{km'}{q} Y^*,$$

$$I_1^* = \frac{\alpha a_8 - bS_1^*}{2a_1},$$

$$I_1^* > 0, \text{ if } a\left(mX^* + m'Y^*\right) > bS_1^*,$$

$$I_2^* = \frac{(1-\alpha) a_8 - bS_2^*}{b'},$$

$$I_2^* > 0, \text{ if } (1-\alpha)\left(mX^* + m'Y^*\right) > bS_2^*.$$

Now we investigate a criteria for endemic steady state E_1 to exist. Here we need that the system of equations (3.1) should have a positive solution. From the set of equation (3.1), we get

$$X^* = b_1 - b_2 Y^*, \tag{3.2}$$

which implies

$X^* \to b_1$ when $Y^* \to 0$ and $Y^* \to b_1 / b_2$ when $X^* \to 0$.

We can also obtain $\dfrac{dX^*}{dY^*} = -b_2$,

$\dfrac{dX^*}{dY^*} < 0$, if $(m' + qb')b > pb'm'$.

Hence, we find that X^* is decreasing function of Y^*

Also, form the set of equations (3.1) we get

$$a_{10}X^* + \left[2ka_{12} - (a_{14} + a_{13})\right]Y^* + 2a_{15} = \left[\left\{\alpha a_{10}X^* + (\alpha a_{14} - ka_{12})Y^* - a_{15}\right\}^2 + b' a_{11} a_{10} X^*\right]^{1/2}$$

$$+\left[\left\{(1-\alpha)a_{10}X^* + ((1-\alpha)a_{14} - ka_{12})Y^* - a_{15}\right\}^2 + b' a_{10} a_{16} X^*\right], \tag{3.3}$$

which implies

$$a_{22}(X^*)^4 + a_{23}(X^*)^3 + a_{24}(X^*)^2 + a_{25}X^* + a_{26} = 0, \text{ when } Y^* = 0. \tag{3.4}$$

By Discarte's rule of sign the equation (3.4) has at least one positive root under condition $2bb' > 1$. Let us denote that positive root by Q_1.

Hence, $X^* \to Q_1$ when $Y^* \to 0$.

Further, when $X^* \to 0$, we get

$$a_{27}(Y^*)^4 + a_{28}(Y^*)^3 + a_{29}(Y^*)^2 + a_{30}Y^* + a_{31} = 0. \tag{3.5}$$

By Discarte's rules of sign, the equation (3.5) also has at least one positive root under condition $2bb' > 1$. Let us denote that positive root by Q_2.

Hence $Y^* \to Q_2$ when $X^* \to 0$.

We can also obtain

$$\frac{dX^*}{dY^*} = \frac{\left(\dfrac{\alpha a_{33}}{a_{32}^{1/2}} + \dfrac{(1-\alpha)a_{35}}{a_{34}^{1/2}}\right)a_{14} - \left\{\left(\dfrac{a_{33}}{a_{32}^{1/2}} + \dfrac{a_{35}}{a_{34}^{1/2}}\right)ka_{12} + 2ka_{12} - (a_{14} + a_{13})\right\}}{a_{10} - \left\{\left(\dfrac{\alpha a_{33}}{a_{32}^{1/2}} + \dfrac{(1-\alpha)a_{35}}{a_{34}^{1/2}}\right)a_{10} + \dfrac{1}{2}\left(\dfrac{a_{11}}{a_{32}^{1/2}} + \dfrac{a_{16}}{a_{34}^{1/2}}\right)b' a_1\right.}.$$

$$\frac{dX^*}{dY^*} < 0 \text{, if } \left(\frac{\alpha a_{33}}{a_{32}^{1/2}} + \frac{(1-\alpha)a_{35}}{a_{34}^{1/2}}\right)a_{14} < \left\{\left(\frac{a_{33}}{a_{32}^{1/2}} + \frac{a_{35}}{a_{34}^{1/2}} + 2\right)ka_{12} - (a_{14} + a_{13})\right\},$$

$$\left(\frac{a_{33}}{a_{32}^{1/2}} + \frac{a_{35}}{a_{34}^{1/2}} + 2\right)ka_{12} > (a_{14} + a_{13})$$

and

$$0 < \left[\frac{\alpha a_{33}}{a_{32}^{1/2}} + \frac{(1-\alpha)a_{35}}{a_{34}^{1/2}} + \frac{1}{2}\left(\frac{a_{11}}{a_{32}^{1/2}} + \frac{a_{16}}{a_{34}^{1/2}}\right)b'\right] < 1$$

or if

$$\left(\frac{\alpha a_{33}}{a_{32}^{1/2}} + \frac{(1-\alpha)a_{35}}{a_{34}^{1/2}}\right)a_{14} > \left\{\left(\frac{a_{33}}{a_{32}^{1/2}} + \frac{a_{35}}{a_{34}^{1/2}} + 2\right)ka_{12} - (a_{14} + a_{13})\right\},$$

$$\left(\frac{a_{33}}{a_{32}^{1/2}} + \frac{a_{35}}{a_{34}^{1/2}} + 2\right)ka_{12} > (a_{14} + a_{13})$$

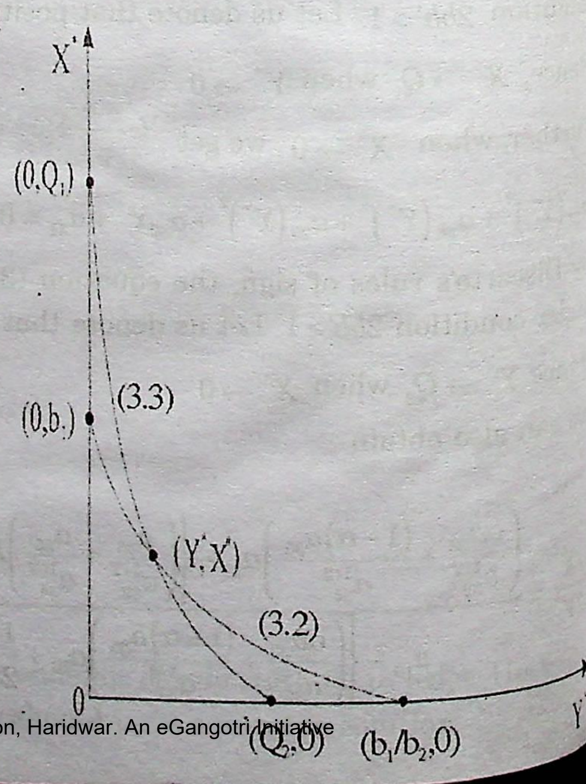
and

$$\left[\frac{\alpha a_{33}}{a_{32}^{1/2}} + \frac{(1-\alpha)a_{35}}{a_{34}^{1/2}} + \frac{1}{2}\left(\frac{a_{11}}{a_{32}^{1/2}} + \frac{a_{16}}{a_{34}^{1/2}}\right)b'\right] > 1$$

Hence, we find that X^* is decreasing function of Y^*. Form above, we see that the two isoclines given by (3.2) and (3.3) will intersect provided that $Q_1 > b_1$ and $\frac{b_1}{b_2} > Q_2$.

Hence, with the above considerations, equation (3.2) represents X^* as decreasing from b_1 and equation (3.3) represents X^* as decreasing from Q_1.

Therefore, two isoclines must intersect provided that intersection value $0 < X^* \leq Q_1$ and $0 < Y^* \leq b_1/b_2$,



exists in the positive Y^*–X^* plane, shown in Fig. 2.

Note : The values of constants a_i $(i=1,2,...,35)$ and b_i $(i=1,2)$ are given in the Appendix.

4. Qualitative Analysis. Now to determine local and global dynamics of steady states E_0 and E_1 of system (2.1) we use, Lyapunov's Second Method. We analyze local dynamic of above steady states (E_0 and E_1) and therefore find sufficient conditions under which E_0 and E_1 are locally asymptotically stable in the form of the following **Theorems** 4.1 and 4.2 respectively:

Theorem 4.1. Let the following inequalities hold

$$b' > \left(kS_1^\circ + \alpha m'\right)/2 \tag{4.1a}$$

$$b > \left\{kS_2^\circ + (1-\alpha)m\right\} \tag{4.1b}$$

$$b' > \left(kS_2^\circ + (1-\alpha)m'\right)/2 \tag{4.1c}$$

$$m > (b + pb') \tag{4.1d}$$

$$m' > qb'. \tag{4.1e}$$

Then E_0 is locally asymptotically stable.

Proof. Using the following positive definite function in the linearized form of system (2.1),

$$V_1 = \left[n_1^2 + A_1 n_2^2 + A_2 n_3^2 + A_3 n_4^2 + A_4 n_5^2 + A_5 n_6^2\right]/2, \tag{4.2}$$

Where, A_i are arbitrary positive constants $(i=1,2,...,6)$, a it can be checked that the derivative of V_1 with respect to t under the conditions (4.1) is negative definite. Hence in view of theory of stability, E_0 is locally asympotically stable.

Theorem 4.2. Steady state E_1 of the system (2.1) is locally asymptotically stable if

$$b' > \left(kS_1^* + kI_2^* + \alpha m'\right)/2, \tag{4.3a}$$

$$\left(kI_1^\circ + b\right) > \left[kS_2^* + (1-\alpha)m\right]/2, \tag{4.3b}$$

$$b' > \left(kS_2^* + kI_1^\circ + (1-\alpha)m'\right)/2, \tag{4.3c}$$

$$m > (b + pb'), \tag{4.3d}$$

$$m' > qb', \tag{4.3e}$$

and satisfied

Proof. Similar to Theorem 4.1.

Now to show that steady states E_0 and E_1 of system (2.1) are globally asymptotically stable, we consider a region of attraction for the system (2.1) in the form of following

Lemma 1. Consider the set

$$R_1 = \{(S_1, I_1, S_2, I_2, X, Y) : 0 < S_{1m} \le S_1 \le P, 0 < I_{1m} \le I_1 \le P, 0 < S_{2m} \le S_2 \le P,$$

$$0 < I_{2m} \le I_2 \le P, 0 \le x, 0 \le Y$$

Which is a region of attraction for all solutions initially in the positive orthant, where $S_{1m} = S_{2m} = I_{1m} = I_{2m}$ are positive constants.

Now we obtain here the conditions for asymptotic stabiity of positive steady states E_0 and E_1 of system (2.1) in non-linear (global) case in the form of following

Theorem 4.3 . The steady state E_0 of system (2.1) is non-linearly (globally) asmptotically stable in a region R_1 given by Lemma 1 where the following conditions are satisfied :

$$0 < B_4 < 2, \tag{4.5a}$$

$$b' > (kP + \alpha m')/2, \tag{4.5b}$$

$$b > \{kS_{2m} + (1-\alpha)m\}/2, \tag{4.5c}$$

$$b' > \{kP + (1-\alpha)m'\}/2, \tag{4.5d}$$

$$m > (b + pb'), \tag{4.5e}$$

$$m' > qb', \tag{4.5f}$$

where B_4 is arbitrary positive constant.

Proof. Consider the following positive definite function about E_0

$$V_2 = \left[u_1^2 + B_1 u_2^2 + B_2 u_3^2 + B_3 u_4^2 + B_4 u_5^2 + B_5 u_6^2\right]/2. \tag{4.6}$$

Here

B_i are arbitrary positive constants (i=1,2,...,6).

Take perturbations in $E_0\left(S_1^\circ, I_1^\circ, S_2^\circ, I_2^\circ, X^\circ, Y^\circ\right)$ as $u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)$ and $u_6(t)$ respectively and putting

$$S_1 = S_1^\circ + u_1(t), I_1 = I_1^\circ + u_2(t),$$

$$S_2 = S_2^\circ + u_3(t), I_2 = I_2^\circ + u_4(t),$$

$X = X^{*} + u_5(t), Y = Y^{*} + u_6(t)$,

in the system (2.1), we get

$$\frac{du_1}{dt} = -bu_1 - k\left(S_1^{*} + u_1\right)u_4 + \alpha m u_5 , \tag{4.7a}$$

$$\frac{du_2}{dt} = -b' u_2 + k\left(S_1^{*} + u_1\right)u_4 + \alpha m' u_6 , \tag{4.7b}$$

$$\frac{du_3}{dt} = -k\left(S_2^{*} + u_3\right)u_2 - bu_3 + (1-\alpha)mu_5 , \tag{4.7c}$$

$$\frac{du_4}{dt} = k\left(S_2^{*} + u_3\right)u_2 - b' u_4 + (1-\alpha)m' u_6 , \tag{4.7d}$$

$$\frac{du_5}{dt} = bu_1 + pb' u_2 + bu_3 + pb' u_4 - mu_5 , \tag{4.7e}$$

$$\frac{du_6}{dt} = qb' u_2 + qb' u_4 - m' u_6 . \tag{4.7f}$$

D i f f e V_2 r with respect to t along the solution of (2.1) and using Lemma 1, we get

$$\frac{dV_2}{dt} = -bu_1^2 - b' B_1 u_1^2 - bB_2 u_3^2 - b' B_3 u_4^2 - mB_4 u_5^2 - m' B_5 u_6^2 - kS_{1m} u_1 u_4 + (\alpha m + bB_4) u_1 u_5$$
$$+ kPB_1 u_2 u_4 + \alpha m' B_1 u_2 u_6 - kS_{2m} B_2 u_2 u_3 + \left[(1-\alpha)mB_2 + bB_4\right]u_3 u_5 + kPB_3 u_2 u_4$$
$$+ (1-\alpha)m' B_3 u_4 u_6 + pb' B_4 u_2 u_5 + pb' B_4 u_4 u_5 + qb' B_5 u_2 u_6 + qb' B_5 u_4 u_6 . \tag{4.8}$$

Applying the inequality $\pm ab \le \left(a^2 + b^2\right)/2$ and making algebraic manipulation, we derive

$$\frac{dV_2}{dt} \le -\left[B_{11}u_1^2 + B_{12}u_2^2 + B_{13}u_3^2 + B_{14}u_4^2 + B_{15}u_5^2 + B_{16}u_6^2\right], \tag{4.9}$$

where

$B_{11} = b - (kS_{1m} + \alpha m + bB_4)/2,$

$B_{12} = b' B_1 - (kPB_1 + kS_{2m} B_2 + \alpha m' B_1 + kPB_3 + pb' B_4 + qb' B_5)/2,$

$B_{13} = bB_2 - \left[kS_{2m}B_2 + (1-\alpha)mB_2 + bB_4\right]/2.$

$$B_{14} = b'B_3 - \left[kS_{1m} + kPB_1 + kPB_3 + (1-\alpha)m'B_3 + qb'B_5 + pb'B_4\right]/2,$$

$$B_{15} = mB_4 - \left\{\alpha m + (1-\alpha)mB_2 + 2(b+pb')B_4\right\}/2,$$

$$B_{16} = m'B_5 - \left[\alpha m'B_1 + (1-\alpha)m'B_3 + 2qb'B_5\right]/2.$$

From (4.9), it can be shown that, $\frac{dV_2}{dt}$ is negative definite under following conditions

$$b > (kS_{1m} + \alpha m + bB_4)/2, \tag{4.10a}$$

$$b'B_1 > \left[kPB_1 + kS_{2m}B_2 + kPB_3 + \alpha m'B_1 + qb'B_5 + pb'B\right], \tag{4.10b}$$

$$bB_2 > \left[kS_{2m}B_2 + (1-\alpha)mB_2 + bB_4\right]/2, \tag{4.10c}$$

$$b'B_3 > \left[kS_{1m} + kPB_1 + kPB_3 + (1-\alpha)m'B_3 + qb'B_5 + pb'B_4\right]/2, \tag{4.10d}$$

$$mB_4 > \left[\alpha m + 2bB_4 + 2pb'B_4 + (1-\alpha)mB_2\right]/2, \tag{4.10e}$$

$$m'B_5 > \left\{\alpha m'B_1 + (1-\alpha)m'B_3 + 2qb'B_5\right\}/2. \tag{4.10f}$$

The above sufficient conditions for $\frac{dV_2}{dt}$ to be negative dfinite, may be further manipulated to get the following simplified conditions:

In (4.10a) choosing B_4 as

$$0 < B_4 < 2, \qquad \text{...(4.11a)}$$

and in (4.10b) choosing B_1 as

$$B_1 > \frac{kS_{2m}B_2 + kPB_3 + pb'B_4 + qb'B_5}{b' - (kP + \alpha m')/2},$$

the condition (4.10b) reduces to

$$b' > (kP + \alpha m')/2. \tag{4.11b}$$

In (4.10c) choosing B_2 as

$$B_2 > \frac{(bB_4)/2}{b - \left\{kS_{2m} + (1-\alpha)m\right\}/2},$$

the condition (4.10c) reduces to

$$b > \left\{kS_{2m} + (1-\alpha)m\right\}/2. \tag{4.11c}$$

In (4.10d) choosing B_3 as

$$B_3 > \frac{\left(kS_{1m} + kPB_1 + pb'B_4 + qb'B_5\right)}{b' - \left(kP + (1-\alpha)m'\right)/2},$$

the condition (4.10d) reduces to

$$b' > \left(kP + (1-\alpha)m'\right)/2 \ . \tag{4.11d}$$

In (4.10e) choosing B_4 as

$$B_4 > \frac{\left(\alpha m + (1-\alpha)mB_2\right)/2}{m - (b + pb')} \ ,$$

the condition (4.10e) reduces to

$$m > (b + pb') . \tag{4.11e}$$

In (4.10f) choosing B_5 as

$$B_5 > \frac{\left(\alpha m' B_1 + (1-\alpha)B_3\right)/2}{2m' - 2qb'},$$

the condition (4.10f) reduces to

$$m' > qb' . \tag{4.11f}$$

Hence, disease free steady state E_0 of system (2.1) is non-linearly (Globally) asymptotically stable in the region R_1 under the conditions given by (4.11), provig Theorem 4.3.

Similarly we also determine that steady state E_1 of system (2.1) is also globally asymptotically stable in a region R_1 given by Lemma 1 under the following conditions :

$$kI_{2m} + b > \left(\alpha m + kPD_1 + kS_1^* + bD_4\right)/2, \tag{4.12a}$$

$$b' > \left(kS_2^* + kP + \alpha m'\right)/2, \tag{4.12b}$$

$$\left(kI_1^* + b\right) > \left(kS_{2m} + (1-\alpha)m\right)/2, \tag{4.12c}$$

$$b' > \left\{kI_1^* + kp + (1-\alpha)m'\right\}/2, \tag{4.12d}$$

$$m > (b + pb'), \tag{4.12e}$$

$$m' > qb', \tag{4.12f}$$

where D_1 and D_4 are arbitrary positive constants.

5. Conclusion. In this model, disease free and endemic steady states, have

been obtained, which are shown to be both linearly (locally) and non-linearly (globally) asymptotically stable under the conditions involving disease related parameters.

From the qualitative analysis of the disease free steady state E_0, it may be concluded that the infection will die out eventually in the underlying population and only mature and immature population of susceptible males and females will exist. From the qualitative analysis of the endemic steady state E_1, it may be concluded that the infection will remain always in the population provided the following disease related parameters satisfy

$$\alpha\left(mX^* + m'Y^*\right) > bS_1^*, (1-\alpha)\left(mX^* + m'Y^*\right) > bS_2^*, 2bb' > 1, m > (b + pb') \text{ and } m' > qb'.$$

APPENDIX

$$a_1 = kb,\ a_2 = \alpha a_{10}X^* + (\alpha a_{14} - ka_{12})Y^* - a_{15},\ a_3 = \alpha\left(a_6X^* + a_7Y^*\right),$$

$$a_4 = (1-\alpha)a_{10}X^* + \left[(1-\alpha)a_{14} - ka_{12}\right]Y^* - a_{15},\ a_5 = (1-\alpha)\left(a_6X^* + a_7Y^*\right)$$

$$a_6 = mb', a_7 = m', a_8 = mX^* + m'Y^*, a_{10} = km, a_{11} = 4b\alpha, a_{12} = \frac{m'}{q}, a_{13} = 4kpb', a_{14}km',$$

$$a_{15} = bb', a_{16} = 4b(1-\alpha), a_{17} = a_{14} + a_{13} - 2ka_{12}, a_{18} = (1-\alpha)a_{14} - ka_{12}, a_{19} = \alpha a_{14} - ka_{12},$$

$$a_{20} = a_{14}a_{16} - 2(1-\alpha)a_{14}a_{15} + 2ka_{12}a_{15}, a_{21} = a_{14}a_{11} - 2\alpha a_{14}a_{15} + 2ka_{12}a_{15},$$

$$a_{22} = \left[\left\{\alpha^4 + (1-\alpha)^4 - 2\alpha^2(1-\alpha)^2\right\}a_{10}^4 - 2\left\{\alpha^2 + \left(1-\alpha^2\right)\right\}a_{10}^3 + a_{10}^2\right],$$

$$a_{23} = \left[\alpha^3 + (1-\alpha)^3 - \left\{\alpha(1-\alpha) + 5/2\right\}\right]4a_{10}^3a_{15} + \left\{\alpha^2 + (1-\alpha)^2\right\}a_{10}^2 + 8a_{10}a_{15},$$

$$a_{24} = \left\{\alpha^4 + (1-\alpha)^4\right\}a_{10}^4 - 4\left[2\left\{\alpha^2 + (1-\alpha)^2\right\} - 5\right]a_{10}^2a_{15},$$

$$a_{25} = 2\left\{\alpha^2 + (1-\alpha)^2\right\}a_{10}^2a_{15}^2 - 60a_{10}a_{15}^3, a_{26} = 4a_{15}^2\left(1 - 4a_{15}^2\right),$$

$$a_{27} = a_{17}^4 + a_{18}^4 + a_{19}^4 + 2a_{18}^2a_{19}^2 - 2a_{17}^4\left(a_{18}^2 + a_{19}^2\right),$$

$$a_{28} = 2a_{18}^2a_{20} + 2a_{19}^2a_{21} - 8a_{17}^2a_{15} - 2\left[a_{19}^2a_{20} + a_{18}^2a_{21} + a_{17}^2\left(a_{20} + a_{21}\right) - 4a_{17}a_{15}\left(a_{18}^2 + a_{19}^2\right)\right]$$

$$a_{29} = a_{18}^4 + a_{19}^4 + 2a_{18}^2a_{15}^2 + 2a_{19}^2a_{15}^2 + 24a_{17}^2a_{15}^2 - 2\left[a_{19}^2a_{15}^2 + a_{20}a_{21} + a_{15}^2a_{18}^2 + 4a_{15}^2\left(a_{18}^2 + a_{19}^2\right)\right.$$

$$\left. -4a_{17}a_{15}\left(a_{20} + a_{21}\right) - 2a_{17}^2a_{15}^2\right],$$

$$a_{30} = 2a_{18}^2 a_{15}^2 + 2a_{19}^2 a_{15}^2 - 32a_{17}a_{15}^3 - 2\left[a_{21}a_{15}^2 + a_{20}a_{15}^2 + 4a_{15}^2(a_{20} + a_{21}) - 8a_{17}a_{15}^3\right],$$

$$a_{31} = 4a_{15}^2\left(1 - 4a_{15}^2\right), a_{32} = a_{33}^2 + b'a_{10}a_{11}X^*, \quad a_{33} = \alpha a_{10}X^* + (\alpha a_{14} - ka_{12})Y^* - a_{15},$$

$$a_{34} = a_{35}^2 + b'a_{10}a_{16}X^*, a_{35} = (1-\alpha)a_{10}X^* + \{(1-\alpha)a_{14} - ka_{12}\}Y^* - a_{15},$$

$$b_1 = \frac{Pb}{m+b}, b_2 = \frac{(1 + qb'/m' - pb'/b)m'}{(m+b)qb'/b}.$$

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ON UNIT OF LENGTH MEASUREMENT IN THE INDUS-SARASWATI CIVILIZATION AND SPEED OF LIGHT IN THE VEDIC LITERATURE

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ABSTRACT

An exact value of the unit of length measurement used in Indus-Saraswati Civilization, has been determined from the precise scale discovered by Ernest Mackay in the 1930-31 season excavation at Mohenjodaro and further correlated with the present day units of measurement. It has been calculated and shown that the speed of light as given in the Vedic literture, when referenced with this erstwhile unit of length measurement (used in Indus-Saraswati Civilization), works out to be precisely equal to the speed of light as per modern measurements. The present paper is a step-wise process followed to unveil this equality.

2010 Mathematics Subject Classification : Primary 01AXX Secondary 01A32

Keywords: Unit of length measurement, Indus-saraswati civilization, speed of light in vedic literature.

1. The Precise Scale. In his 1930-31 season at Mohanjo-daro, Ernest Mackay discovered a broken piece of shell bearing 8 divisions of precisely 6.7056mm each, with a dot and circle five graduations apart, which suggests a decimal system. However, attempts by Mackay, to relate such a unit to dimensions in Mohanjo-daro, were not very successful (Michel Danino [2]) and thus were abandoned.

2. Units of Length in Chanakya's Arthashastra. Chanakya was the political of the legendary monarch Chandragupta Maurya of 4th century BC. He was aman learned in may disciplines and wrote the famous treatise on economics called the Arthashastra-meaning the books of money). In Arthashastra, Chanakya mentions two types of *Dhanushas* as units for measuring length and distances. One is the ordinary *Dhanusha,* consisting of 96 *Angulas,* and the other *Dhanusha* is mentioned as *Garhpatya Dhanusha* and consists of 108 *Angulas.* Chanakya also mentions many other units including a *Dhanurgraha,* which consists of 4 *Angulas* and a *Yojana* (In Sanskrit-English dictionary, by Moniere William, a Yojana has been defined as a mesure of distance=4 Kosas or about 9 miles. Apart from the Moniere William dictionary, several other books/sources also give the Yojana as eqal to about 9 miles.), as consisting of 8000 *Dhanushas.*

3. Decoding the Mohan-jo-daro Scale. If we keep 10 divisions of the Mohanjo-daro scale as equal to a *Dhanurgraha* or 4 *Angulas*, the precise length of an *Angula* works out to be 16.764*mm*.

A *Dhanusha* of 96 *Angulas* $=96\times16.764\text{mm}=1.609344\text{m}$ (1)

and

A *Dhanusha* of 108 *Angulas* $= 108\times16.764mm = 1.810512 meters.$ (2)

1 *Yojana* $=8000$ *Dhanushas* (of 108 *Angulas* each) (3)

Thus

1 *Yojana* $=8000\times1.810512\text{m}=14.484096km.$ (4)

Further

$14.484096km=9$ *miles*, (*exactly!*). (5)

Also

1000 *Dhanushas of 96 Angulas* each $=1mile$ (6)

Interestingly, when we look into the history of *mile,* we find that the word mile is derived from *mile,* which means a *thousand.* This points to universal adaptation of ancient units of length.

4. Corroboration from Other Sources

The Indus Inch: The Indus civilisation unit of length, widely known as *Indus Inch* was 1.32 Inches which is exactly equal to 2 Angulas of 16.76mm each.

Mohenjo-daro's Great Bath: The height of the corbelled drain forming the outlet of Mohenjo-daro's Great Bath [6, pp. 133-142] is about 1.8m, which is equal to a Dhanusha of 108 Angulas of 16.764mm each.

Standard Street-Widths. Kalibangam, a city in the Indus-Saraswati Civilization (in Rajasthan) had street widths [8] of 1.8m, 3.6m, 5.4m and 7.2m i.e. built to the standard dimensions being equal to 1 Dhanusha, 2 Dhanushas, 3 Dhanushas and 4 Dhanushas respectively. Such widths are found at other sites also. Bigger streets of Banawali [8] another town in Indus-Saraswati Civilization (in Haryana) measure 5.4m i.e. they were built with the unit of 3 Dhanushas.

Taj Mahal. A Persian manuscript "*Shah Jahan Nama*" contains a very particular description of three principal buildings of Agra-the Taj Mahal, Moti Masjid and Jamah Masjid. In the "*Shah Jahan Nama*", the dimensions of these three buildings are given in *Gaz*. These dimensions were got measured by col. J.A. Hodgson in December, 1825, in feet and inches. The various (28) dimensions, in feet and inches, as well as in *Gaz,* are given by Hodgson in his article [5], in *Table A*. He has also given *Table B* (25 dimensions) excluding 3 dimensions which he thought were not very dependable. The weighted mean length of a *Gaz* works out to 31.70 inches, (80.52cm) from *Table A* and 31.66 inches=80.42cm from *Table B*. The average of the two values in 31.68 inches=80.47cm. It is pertinent to mention here that Barraud [1, pp. 108-109, 258-259] in The complete Taj Mahal and the River f

Columns of Agra, has taken a *Gaz* as equal to 80.5cm which is very nearly equal to 80.47cm as worked out above. Taking a Dhanusha of 96 Angulas to be equal to 2 *Gaz*, the length of an Angula works out to 16.764mm. It shows that a *Gaz* of 48 Indus-Saraswati Angulas was being used, even in the days of Shah Jahan's rule i.e. 17th Century A.D.

The Gudea's Rule. The *Gudea's* rule (2175 B.C.) preserved in the Louvre shows intervals in Sumerian Shusi of 0.66 inches, which is exactly equal to the Indus-Saraswati Angula of 16.764mm.

Mayan Units of Measurement. Drewitt [3] and Drucker [4] made a study of the ancient city of Teotihuacan, belonging to *Mayan* Civilization, in Mexico, and hypothesised a unit of 80.5cm, which is very nearly equal to the Indian *Guz* of 48 Angulas=80.47cm.

Temple Wall-Engravings. Nearer home, two engravings on a wall of the temple at Tiruputtkali (12th Century A.D.) near Kanchipuram, show two scales [7] one measuring 7.24 metres in length, with marketing dividing the scale into 4 equal parts, and the second one measuring 5.69 metres in length and marking dividing the scale into 4 equal parts. It may be observed that each division of the first scale is precisely equal to a Dhanusha of 108 Angulas of 16.764mm each. Interestingly, the second scale is precisely equal to π times Dhanusha i.e. equal to the circumference of a circle with one Dhanusha as its Diameter.

It is interesting to note here that Mackay reports [9] at Mohenjo-daro, a lane and a doorway having both a width of 1.42m, which is precisely equal to one division of the second scale at the Tiruputtkali Temple, indicating that both the scales were prevalent in Indus-Saraswati Civilization as well as in South India.

It proves beyond doubt that the Units of measurement as derived from the precise scale found at Mohenjo-daro were prevalent not only in the Indus-Saraswati Civilization, but also in South India and in the ancient *Sumerian, Egyption and Mayan Civilizations.*

5. Speed of Light in Vedic Literature. In the commentary on Rig-Veda. Mandal 1, Sukta 50, Mantra 4, which is in praise of the Sun god, Sayanacharya (14th Century AD) writes:

तथा च स्मर्यते...

योजनानां सहस्रं द्वे द्वे शते द्वे च योजने।

एकेन निमिषार्धेन क्रममाण नमोस्तु ते।।

Meaning... "It is remembered that...

Salutations to Thee (the Sun) who approacheth (at a speed of) 2202 yojanas in a nimishardha (*half nimisha*)."

Clearly it is the Speed of light (or sunrays) that is mentioned in the *Shloka*.

This *shloka* is attributed [10, pp. 67] to the son of Kanva Maharshi (4000 B.C.). *Bhatta Bhaskara* (10th Century) mentions [11] this *shlok* in his commentary on the *Taitreya Brahamana*.

To put it in mathematical terms, as stated by the above shlok, the speed of light would be :

$$\textit{Speed of Light} = 2202 \text{ Yojanas/Nimishardha} \tag{7}$$

We have already calculated the value of Yojana in modern unit of km in the previous step (Equation 4), but we still do not know, what the *Nimishardha* translates into. Let us dig into another Vedic text the *Vishnu Puran* to find how the erstwhile units of time can be related to the modern units of time.

In the Vishnu Puran (Book 1, Chapter 3, Shloka 8,9), it is stated that:

15 *Nimishas*	= 1 *Kashtha*
30 *Kashthas*	= 1 *Kala*
30 *Kalas*	= 1 *Mahurta*
30 *Mahurtas*	= 1 *day* and *night* (अहोरात्रम्)

Thus

one *day* and *night* = 405,000 *Nimishas*=810,000 *Nimishardhas*. (8)

(Literal meaning of Nimishardha being half of Nimisha).

In *Surya Sidhant* (Chapter 1, Shloka 12), it is mentioned that 60 Nadis constitute one *Sidereal* Day and Night (नाक्षत्रम् अहोरात्रम्). It is also well known that 1 *Mahurta*=2 *Ghatis* or 2 *Nadis*. It is clear from this that in Astronomical calculations, the sidereal day was taken as the unit of time. A sidereal day is the time taken by the stellar constellations to complete one revolution around the Earth. A sidereal day is equal to 23 hours, 56 minutes and 4.1 seconds or equivalent sec (Wikipedia [12]).

Thus

$$\textit{Nimishardha} = 86164.1/810000 \text{ sec} = 0.1063754\text{sec} \tag{9}$$

The speed of light as given in the Vedic Literature therefor comes out to be;

$$\frac{2202 * 14.484096}{0.1063754}\frac{km}{\text{sec}} = 2.998 * 10^5 \, km/\text{sec}. \tag{10}$$

which is precisely equal to the speed of light as per modern measurements.

6. Conclusions. The calculations and references int he article endeavor to establish the vast reach of the Indus Valley scholars, not only academically, but also geographically. Thousands of years before the modern scientists rediscovered the speed of light; our ancestors knew of the exact same value and referred to it as an exalted property of the Sun-God to salute Him. It is also heartening to know that the modern-age concept of globalization and sharing of ideas between cultures

was also an established way of life in their era, as proven by the shared units of measurement found across far-flung cultures. When and how these ancient civilization lost their influence is still a mystery, but for now we can celebrate the knowledge that :

1. The Speed of Light as given in the Vedic literature, is precisely equal to the Speed of Light as per modern measurements.

2. The basic unit of length measurement in the Indus-Saraswati Civilization was an *Angul* of 16.764mm. This unit was used not only in the Indus-Saraswati Civilization, but also in South India, and other ancient world Civilizations including *Sumerian, Egyptian* and *Mayan* Civilizations.

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